

Large deviations for the exclusion process with a slow bond

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Abstract: We consider the one-dimensional symmetric simple exclusion process with a slow bond. In this model, whilst all the transition rates are equal to one, a particular bond, the *slow bond*, has associated transition rate of value N^{-1} , where N is the scaling parameter. This model has been considered in previous works on the subject of hydrodynamic limit and fluctuations. In this paper, assuming uniqueness for weak solutions of hydrodynamic equation associated to the perturbed process, we obtain dynamical large deviations estimates in the diffusive scaling. The main challenge here is the fact that the presence of the slow bond gives rise to Robin's boundary conditions in the *continuum*, substantially complicating the large deviations scenario.

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1. Introduction

In this paper we present dynamical large deviations estimates for the Symmetric Simple Exclusion Process (SSEP) with a slow bond. The SSEP is a largely studied process both in Probability and Statistical Mechanics. It consists of particles that perform independent random walks in a certain graph, except for the exclusion rule that prevents two or more particles from occupying the same site.

The SSEP with a *slow bond* is characterized by a defect at a fixed bond. The graph here considered is $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$, the discrete one-dimensional torus with N sites. Let us describe this process in terms of clocks. At each bond we associate a different Poisson clock, all of them independent. When a clock rings, the occupation at the sites connected by the corresponding bond are exchanged. Of course, if both sites are empty or occupied, nothing happens. We call the parameters of those Poisson clocks of *exchange rates*. All exchange rates are equal to one, except at the slow bond which has exchange rate N^{-1} , which slows down the passage of particles there. Notice that the choice of the exchange rates characterizes the non-homogeneity of the environment.

This model has origin in the models considered in [FJL, FL]. In [FJL], the exchange rate at a bond of vertices x and $x + 1$ is taken as $[N(W(x + 1/N) - W(x/N))]^{-1}$, where W is a α -stable subordinator of a Lévy process. In the same line, [FL] dealt with exchange rates driven by a general, non-random, strictly increasing function W . The SSEP with a slow bond is in fact a particular case of the model considered in [FL].

In order to understand the collective behavior of the microscopic system, a natural question is the limit for the time evolution of the spatial density of particles, usually called *hydrodynamic limit*, see [KL] and references therein. The limiting density of a given system is usually characterized as the weak solution of some partial differential equation, being the associated equation denominated *hydrodynamic equation*.

By [FL, FGN1, FGN2], the hydrodynamic limit of the SSEP with a slow bond is well understood, being the hydrodynamic equation given by following heat equation with Robin's boundary conditions:

$$\begin{cases} \partial_t \rho(t, u) = \partial_u^2 \rho(t, u), & t > 0, u \in \mathbb{T} \setminus \{0\}, \\ \partial_u \rho(t, 0^+) = \partial_u \rho(t, 0^-) = \rho(t, 0^+) - \rho(t, 0^-), & t > 0, \\ \rho(0, u) = \gamma(u), & u \in \mathbb{T}, \end{cases} \quad (1.1)$$

where \mathbb{T} denotes the continuous one-dimensional torus, 0^+ and 0^- denote the side limits around $0 \in \mathbb{T}$ and $\gamma : \mathbb{T} \rightarrow [0, 1]$ is a density profile. The boundary condition above can be interpreted as the *Fick's Law*: the rate in which mass is exchanged between two media is proportional to the difference of concentration in each medium.

The natural questions that emerge in the sequence are fluctuations and large deviations with respect to the expected limit. Equilibrium fluctuations for the SSEP with a slow bond has been studied in [FGN3]. In this work we analyze the corresponding large deviations, consisting in the occurrence rate of events differing from the expected limit in the scaling of the hydrodynamic limit. The large deviations of a Markov process comes from two origins. One part are deviations from the initial measure, and the second are deviations from the dynamics. These are called statical and dynamical large deviations, respectively. Since the invariant measures for the dynamics here considered are Bernoulli product measure, for which the large deviations are well known, we will treat only the dynamical large deviations: the system will start from *deterministic* initial configurations associated in some sense (Definition 2.2) to a macroscopic profile.

The main difficulty for establishing large deviations for the SSEP with a slow bond of parameter N^{-1} comes from the fact that the limiting occupations at the vertices of the slow bond depend on time, as we can see in the Robin's boundary conditions above. In important previous papers [BLM] and [FLM], the authors have considered exclusion process with fixed rate of incoming and outgoing particles at the boundaries leading to Dirichlet's boundary conditions, therefore with time independent values at the boundaries.

Here it has been considered a single slow bond. An extension to a finite number of slow bonds (in the setting of [FGN1]) would be straightforward, with no additional obstacles. However, it would carry on the notation and probably would imply a loss of clarity. For this reason we decided to focus in the single slow bond case. What is still far from manageability are the large deviations for the model of [FL], which deals with much stronger spatial non-homogeneity (a dense set of slow bonds is allowed there). This is a very interesting and challenging problem.

An important ingredient in the large deviations proof consists in establishing the law of large numbers for a suitable set of perturbations of the original systems. The family of perturbations we have considered is *the weakly asymmetric exclusion process (WASEP) with a slow bond*. Its hydrodynamic equation is a non-linear diffusive partial differential equation with non-linear Robin's bound-

ary conditions. Assuming uniqueness of weak solutions of this equation, which is a delicate question due to the non-linearity at the boundary, we prove the corresponding hydrodynamic limit. Existence of weak solutions is granted by the tightness of the processes.

The Radon-Nikodym derivative of the perturbed process with respect to the original process naturally leads to the expression of the large deviations rate functional. A difficulty in the proof of the upper bound comes from the fact the Radon-Nikodym derivative obtained is not a function of the empirical measure. To overpass this obstacle, we show that the Radon-Nikodym derivative is superexponentially close to a function of the empirical measure. Moreover, following steps of [BLM, FLM] we define an energy and then proving that trajectories with infinity energy are not relevant in the large deviations regime. Carefully handling this facts together we organize the scenario in order to invoke the Minimax Lemma attaining the upper bound for compact sets. Exponential tightness finally leads to the upper bound for closed sets.

Since the upper bound is achieved via an optimization over perturbations, the rate functional obtained turns to be expressed by a variational expression. On the other hand, for the large deviations lower bound, it is required to find the cheapest perturbation that leads the system to a given profile distinct of the expected limit. In other words, it is necessary to solve the variational expression of the rate function, at least for a sufficiently large class of density profiles. This is precisely what we do in the large deviations lower bound, by means of a proof surprisingly simple. In fact, the proof (of Proposition 6.1) consists essentially in checking that the perturbation H that leads the system to a limit ρ^H is the cheapest one. Indeed, a difficult part of the work was to find the correct class of perturbations for the dynamics and fulfil the technical details.

Then, since the rate functional is convex in a specific sense, by a density argument we extend the lower bound for the class of smooth profiles. The extension for general profiles is a hard problem of convex analysis and illustrates that there is much to be developed in terms of Orlicz Spaces as devices in large deviations schemes. This is subject of future work.

The paper is divided as follows. In Section 2, we introduce notation and state the main results, namely: Theorem 2.10 and Theorem 2.14. In Section 3, we establish the replacement lemma and the energy estimates. In Section 4, we prove the Theorem 2.10. In Section 5, we prove the upper bound. Finally, the lower bound for smooth profiles is presented in the Section 6.

2. Model and statements

Let $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z} = \{0, 1, 2, \dots, N-1\}$ be the one-dimensional discrete torus with N points. In each site of \mathbb{T}_N we allow at most one particle. In other words, we consider configurations of particles $\eta \in \{0, 1\}^{\mathbb{T}_N}$. We say that $\eta(x) = 0$, if the site $x \in \mathbb{T}_N$ is vacant and $\eta(x) = 1$, if the site $x \in \mathbb{T}_N$ is occupied. Notice that $x = 0$ and $x = N$ are the same site. Denote by $\Omega_N = \{0, 1\}^{\mathbb{T}_N}$ this state space.

The exclusion process with a slow bond at the bond of vertices $-1, 0$, which has been considered in [FL, FGN1, FGN2], can be described as follows. To each bond of \mathbb{T}_N we associate a Poisson clock, all of them independent. If the bond is that one of vertices $-1, 0$, the parameter of the Poisson is taken as $1/N$. All the others Poisson clocks have parameter one. When a clock rings, the occupation values of η at the vertices of the associated bond are exchanged. The smaller parameter at the bond of vertices $-1, 0$ slows the passage of particles cross it, from where the name *slow bond*.

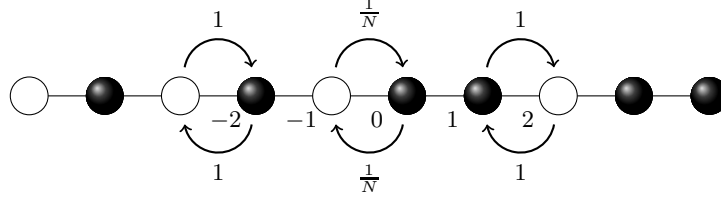


FIG 1. The bond of vertices $\{-1, 0\}$, the slow bond, has particular rates associated to it.

This Markov process can also be characterized in terms of its infinitesimal generator L_N , which acts on functions $f : \Omega_N \rightarrow \mathbb{R}$ as

$$(L_N f)(\eta) = \frac{1}{N} [f(\eta^{-1,0}) - f(\eta)] + \sum_{\substack{x \in \mathbb{T}_N \\ x \neq -1}} [f(\eta^{x,x+1}) - f(\eta)], \quad (2.1)$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the variables $\eta(x)$, $\eta(x+1)$:

$$\eta^{x,x+1}(y) = \begin{cases} \eta(x+1), & \text{if } y = x, \\ \eta(x), & \text{if } y = x+1, \\ \eta(y), & \text{otherwise.} \end{cases} \quad (2.2)$$

Denote by $\{\eta_t; t \geq 0\}$ the Markov process on $\Omega_N = \{0, 1\}^{\mathbb{T}_N}$ associated to the generator L_N , defined in (2.1), *speeded up* by N^2 . The dependency of η_t in N will be omitted to keep notation as simple as possible.

Remark 2.1. This is a notion that often causes confusion and for this reason we explain it in detail. By η_t we mean the Markov process which generator is $N^2 L_N$. Equivalently, η_t could be defined as the Markov process with generator L_N (without the factor N^2) seen at time tN^2 .

Let $\mathcal{D}([0, T], \Omega_N)$ be the path space of càdlàg time trajectories with values in $\Omega_N = \{0, 1\}^{\mathbb{T}_N}$. For short, we will denote this space just by \mathcal{D}_{Ω_N} . Given a measure μ_N on Ω_N , denote by \mathbb{P}_{μ_N} the probability measure on \mathcal{D}_{Ω_N} induced by the initial state μ_N and the Markov process $\{\eta_t^N; t \geq 0\}$. Expectation with respect to \mathbb{P}_{μ_N} will be denoted by \mathbb{E}_{μ_N} . Let ν_α^N be the Bernoulli product measure on Ω_N with marginals given by

$$\nu_\alpha^N \{\eta; \eta(x) = 1\} = \alpha, \quad \forall x \in \mathbb{T}_N.$$

These measures $\{\nu_\alpha^N; 0 \leq \alpha \leq 1\}$ are invariant, in fact reversible, for the dynamics described above. Denote by $\mathbb{T} = [0, 1]$ the one-dimensional continuous torus, where we identify the points 0 and 1.

Definition 2.2. A sequence of probability measures $\{\mu_N; N \geq 1\}$ is said to be associated to a profile $\rho_0 : \mathbb{T} \rightarrow [0, 1]$ if

$$\lim_{N \rightarrow \infty} \mu_N \left[\eta; \left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \eta(x) - \int H(u) \rho_0(u) du \right| > \delta \right] = 0, \quad (2.3)$$

for every $\delta > 0$ and every continuous functions $H : \mathbb{T} \rightarrow \mathbb{R}$.

The quantity introduced in the definition above can be reformulated in terms of empirical measures. We start by defining the set

$$\mathcal{M} = \left\{ \mu; \mu \text{ is a positive measure on } \mathbb{T} \text{ with } \mu(\mathbb{T}) \leq 1 \right\}, \quad (2.4)$$

this space is endowed with the weak topology. Consider the measure $\pi^N \in \mathcal{M}$, which is obtained by rescaling space by N and by assigning mass N^{-1} to each particle:

$$\pi^N(\eta, du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x) \delta_{\frac{x}{N}}(du),$$

where δ_u is the Dirac measure concentrated on u . The measure $\pi^N(\eta, du)$ is called the empirical measure associated to the configuration η . With this notation, $\frac{1}{N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \eta(x)$ is the integral of H with respect to the empirical measure π^N , denoted by $\langle \pi^N, H \rangle$.

We consider the time evolution of the empirical measure π_t^N associated to the Markov process $\{\eta_t; t \geq 0\}$ by:

$$\pi_t^N(du) = \pi^N(\eta_t, du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \delta_{\frac{x}{N}}(du). \quad (2.5)$$

Note that (2.3) is equivalent to say that π_0^N converges in distribution to $\rho_0(u)du$. Throughout the entire paper, it is fixed a time-horizon $T > 0$. Let $\mathcal{D}([0, T], \mathcal{M})$ be the space of \mathcal{M} -valued càdlàg trajectories $\pi : [0, T] \rightarrow \mathcal{M}$ endowed with the Skorohod topology. For short, we will use the notation $\mathcal{D}_{\mathcal{M}} = \mathcal{D}([0, T], \mathcal{M})$. Denote by $\mathbb{Q}_{\mu_N}^N$ the measure on the path space $\mathcal{D}_{\mathcal{M}}$ induced by the measure μ_N and the empirical process π_t^N introduced in (2.5).

2.1. Frequently used notations

Before stating results we present some important notations to be used in the entire paper.

- The indicator function of a set A will be written by $\mathbf{1}_A(u)$, which is one when $u \in A$ and zero otherwise.

- Given a function $H : \mathbb{T} \rightarrow \mathbb{R}$, we will denote $H(0^-)$ and $H(0^+)$, respectively, for the left and right side limits of H at the point $0 \in \mathbb{T}$.
- Given a function $H : \mathbb{T} \rightarrow \mathbb{R}$, denote $\delta H(0) = H(0^+) - H(0^-)$ its jump size at zero. And denote $\delta_N H_x = H(\frac{x+1}{N}) - H(\frac{x}{N})$. Hence, provided H is right continuous at zero, $\delta_N H_{-1}$ converges to $\delta H(0)$.
- Given a function $g : [0, T] \times \mathbb{T}$, we write down $g_t(u)$ to denote $g(t, u)$. It should not be misunderstood with the notation for time derivative, namely $\partial_t g(t, u)$.
- Given a non-negative integer k , denote by $C^k(\mathbb{T})$ the set of real-valued functions with domain \mathbb{T} with continuous derivatives up to order k . As natural, $C(\mathbb{T})$ denotes the set of continuous functions. For non-negative integers j and k , denote by $C^{j,k}([0, T] \times \mathbb{T})$ the set of real valued functions with domain $[0, T] \times \mathbb{T}$ with continuous derivatives up to order j in the first variable (time), and continuous derivatives up to order k in the second variable (space).
- The notation C_k means compact support contained in $[0, T] \times (0, 1)$. For instance, $C_k^{j,k}([0, T] \times (0, 1))$ denotes the subset of $C^{j,k}([0, T] \times (0, 1))$ composed of functions with compact support contained in $[0, T] \times (0, 1)$.
- The notation $g(N) = O(f(N))$ means $g(N)$ is bounded from above by $Cf(N)$, where the constant C does not depend on N . The notation $g(N) = o(f(N))$ means $\lim_{N \rightarrow \infty} g(N)/f(N) = 0$.
- Despite we have denoted $\langle \pi_t^N, H \rangle = \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(\frac{x}{N}) \eta_t(x)$, the bracket $\langle \cdot, \cdot \rangle$ will also mean the inner product in $L^2(\mathbb{T})$ and in $L^2[0, 1]$. The double bracket $\langle\langle \cdot, \cdot \rangle\rangle$ will denote the inner product in $L^2([0, T] \times \mathbb{T})$.

2.2. The hydrodynamic equation

The slow bond, as we will see, yields a discontinuity at the origin in the continuum limit. Therefore, discontinuous functions at the origin are naturally required.

Definition 2.3. Denote by $C^{1,2}([0, T] \times [0, 1])$ the space of functions $H : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ such that

1. H restricted to $[0, T] \times \mathbb{T} \setminus \{0\}$ belongs to $C^{1,2}([0, T] \times \mathbb{T} \setminus \{0\})$;
2. Identifying $\mathbb{T} \setminus \{0\}$ with the open interval $(0, 1)$, H has a $C^{1,2}$ extension to $[0, T] \times [0, 1]$;
3. For any $t \in [0, T]$, H is right continuous at zero, i.e., $H(t, 0) = \lim_{x \rightarrow 0^+} H(t, x)$.

This space of test functions should not be misunderstood with $C^{1,2}([0, T] \times \mathbb{T})$. In words, a function H belongs to this space $C^{1,2}([0, T] \times [0, 1])$ if, “opening” the torus at 0, the function has a $C^{1,2}$ extension to the closed interval $[0, 1]$.

The bracket $\langle \cdot, \cdot \rangle$ will denote indistinctly the inner product in $L^2(\mathbb{T})$ and in $L^2[0, 1]$. Let $L^2([0, T] \times \mathbb{T})$ be the Hilbert space of measurable functions $H : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ such that

$$\int_0^T \int_{\mathbb{T}} (H(s, u))^2 du ds < \infty,$$

endowed with the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ defined by

$$\langle\langle H, G \rangle\rangle = \int_0^T \int_{\mathbb{T}} H(s, u) G(s, u) du ds.$$

Definition 2.4 (Sobolev Space). *Let $\mathcal{H}^1(0, 1)$ be the set of all locally summable functions $\zeta : (0, 1) \rightarrow \mathbb{R}$ such that there exists a function $\partial_u \zeta \in L^2(0, 1)$ satisfying $\langle \partial_u G, \zeta \rangle = -\langle G, \partial_u \zeta \rangle$, for all $G \in C_k^\infty((0, 1))$. For $\zeta \in \mathcal{H}^1(0, 1)$, we define the norm*

$$\|\zeta\|_{\mathcal{H}^1(0,1)} := \left(\|\zeta\|_{L^2(0,1)}^2 + \|\partial_u \zeta\|_{L^2(0,1)}^2 \right)^{1/2}.$$

Let $L^2(0, T; \mathcal{H}^1(0, 1))$ be the space of all measurable functions $\xi : [0, T] \rightarrow \mathcal{H}^1(0, 1)$ such that

$$\|\xi\|_{L^2(0,T;\mathcal{H}^1(0,1))}^2 := \int_0^T \|\xi_t\|_{\mathcal{H}^1(0,1)}^2 dt < \infty.$$

We refer the reader to [Evans, Leoni, TE] for more on Sobolev spaces.

Remark 2.5. An equivalent and useful definition for the Sobolev space $L^2(0, T; \mathcal{H}^1(0, 1))$ is the set of bounded functions $\xi : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ such that there exists a function $\partial \xi \in L^2([0, T] \times \mathbb{T})$ satisfying

$$\langle\langle \partial_u H, \xi \rangle\rangle = -\langle\langle H, \partial \xi \rangle\rangle,$$

for all functions $H \in C_k^{0,1}([0, T] \times (0, 1))$.

Definition 2.6 (The hydrodynamic equation for the SSEP with a slow bond). *Consider a measurable density profile $\gamma : \mathbb{T} \rightarrow [0, 1]$. A function $\rho : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ is said to be a weak solution of the parabolic differential equation with Robin boundary conditions*

$$\begin{cases} \partial_t \rho = \Delta \rho \\ \rho_0(\cdot) = \gamma(\cdot) \\ \partial_u \rho_t(0^+) = \partial_u \rho_t(0^-) = \rho_t(0^+) - \rho_t(0^-), \end{cases} \quad (2.6)$$

if the following two conditions are fulfilled:

$$(1) \rho \in L^2(0, T; \mathcal{H}^1(0, 1));$$

(2) For all functions $G \in C^{1,2}([0, T] \times [0, 1])$ and for all $t \in [0, T]$, ρ satisfies the integral equation

$$\begin{aligned} \langle \rho_t, G_t \rangle - \langle \gamma, G_0 \rangle &= \int_0^t \langle \rho_s, (\partial_s + \Delta) G_s \rangle ds \\ &\quad + \int_0^t \{ \rho_s(0^+) \partial_u G_s(0^+) - \rho_s(0^-) \partial_u G_s(0^-) \} ds \\ &\quad - \int_0^t (\rho_s(0^+) - \rho_s(0^-)) (G_s(0^+) - G_s(0^-)) ds. \end{aligned} \quad (2.7)$$

Assumption (1) guarantees that the boundary integrals are well defined, see [Evans, Leoni] on the notion of trace of Sobolev spaces. The Robin (mixed) boundary conditions in (2.6) can be interpreted as the Fick Law at the point $x = 0$. This is discussed in more detail in [FGN1]. The uniqueness and existence of weak solutions of (2.6) was proved in [FGN2]. Moreover, it was proved in [FL, FGN1, FGN2] that

Theorem 2.7. Fix a measurable density profile $\gamma : \mathbb{T} \rightarrow [0, 1]$ and consider a sequence of probability measures μ_N on Ω_N associated to γ in the sense (2.3). Then, for any $t \in [0, T]$,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu_N} \left[\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \eta_t(x) - \int G(u) \rho_t(u) du \right| > \delta \right] = 0, \quad (2.8)$$

for every $\delta > 0$ and every function $G \in C(\mathbb{T})$. Here, ρ is the unique weak solution of the linear partial differential equation (2.6) with $\rho_0 = \gamma$.

We notice that the result above is a particular case of that considered in [FL], being the characterization in terms of a classical partial differential equation given in [FGN1, FGN2]. Moreover, the statement (2.8) is equivalent to say that π_t^N converges in probability to $\rho_t(u)du$.

2.3. The Weakly Asymmetric Exclusion Process with a slow bond

In order to obtain the Large Deviations of a Markov process, a natural step is to prove the LLN for a class of perturbations of the original Markov process. In our case, the correct perturbations will be given by the class of *weakly asymmetric exclusion processes with a slow bond*, to be defined ahead. For short, we will call it just *WASEP with a slow bond*.

Recall Definition 2.3. Given a function $H \in C^{1,2}([0, T] \times [0, 1])$, consider the time non-homogeneous Markov process whose generator at time t acts on functions $f : \Omega_N \rightarrow \mathbb{R}$ as

$$\begin{aligned} (L_{N,t}^H f)(\eta) &= \sum_{x \in \mathbb{T}_N} \xi_x^N e^{H_t(\frac{x+1}{N}) - H_t(\frac{x}{N})} \eta(x) (1 - \eta(x+1)) \left[f(\eta^{x,x+1}) - f(\eta) \right] \\ &\quad + \sum_{x \in \mathbb{T}_N} \xi_x^N e^{-H_t(\frac{x+1}{N}) + H_t(\frac{x}{N})} \eta(x+1) (1 - \eta(x)) \left[f(\eta^{x,x+1}) - f(\eta) \right], \end{aligned} \quad (2.9)$$

where $\eta^{x,x+1}$ is defined in (2.2) and

$$\xi_x^N = \begin{cases} 1, & \text{if } x \in \mathbb{T}_N \setminus \{-1\}, \\ N^{-1}, & \text{if } x = -1. \end{cases} \quad (2.10)$$

In the particular case H is a constant function, the generator $L_{N,t}^H$ turns out to be equal to the generator L_N defined in (2.1). We emphasize that the asymmetry is weak in all the bonds except at the bond of vertices $-1, 0$. Since the function H is possibly discontinuous at the origin, the asymmetry in that bond does not go to zero in the limit, appearing indeed in the hydrodynamical equation.

Let $\{\eta_t^H; t \geq 0\}$ be the non-homogeneous Markov process with generator $L_{N,t}^H$ defined in (2.9) *speeded up by N^2* . Given a probability measure μ_N on Ω_N , denote by $\mathbb{P}_{\mu_N}^H$ the probability measure on the space of trajectories \mathcal{D}_{Ω_N} induced by the Markov process $\{\eta_t^H; t \geq 0\}$ starting from the measure μ_N .

The empirical measure π_t^N corresponding to $\{\eta_t^H; t \geq 0\}$ is defined in the same way of (2.5). Denote $\chi(\alpha) = \alpha(1 - \alpha)$ the mobility function and $\delta H_t(0) = H_t(0^+) - H_t(0^-)$. Next, we present the hydrodynamic equation for the WASEP with a slow bond.

Definition 2.8. *Let $\gamma : \mathbb{T} \rightarrow \mathbb{R}$ be a bounded density profile and fix $H \in C^{1,2}([0, T] \times [0, 1])$. A function $\rho : [0, T] \times \mathbb{T} \rightarrow [0, 1]$ is said to be a weak solution of the partial differential equation*

$$\begin{cases} \partial_t \rho = \Delta \rho - 2 \partial_u (\chi(\rho) \partial_u H) \\ \rho_0(\cdot) = \gamma(\cdot) \\ \partial_u \rho_t(0^+) = 2 \chi(\rho_t(0^+)) \partial_u H_t(0^+) - \varphi_t(\rho, H), \\ \partial_u \rho_t(0^-) = 2 \chi(\rho_t(0^-)) \partial_u H_t(0^-) - \varphi_t(\rho, H), \end{cases} \quad (2.11)$$

where

$$\varphi_t(\rho, H) = \rho_t(0^-)(1 - \rho_t(0^+)) e^{\delta H_t(0)} - \rho_t(0^+)(1 - \rho_t(0^-)) e^{-\delta H_t(0)}, \quad (2.12)$$

if the following two conditions are fulfilled:

$$(1) \rho \in L^2(0, T; \mathcal{H}^1(0, 1));$$

(2) For all functions G in $C^{1,2}([0, T] \times [0, 1])$, and all $t \in [0, T]$, ρ satisfies the integral equation

$$\begin{aligned} \langle \rho_t, G_t \rangle - \langle \gamma, G_0 \rangle &= \int_0^t \langle \rho_s, (\partial_s + \Delta) G_s \rangle ds + 2 \int_0^t \langle \chi(\rho_s) \partial_u H_s, \partial_u G_s \rangle ds \\ &+ \int_0^t \{ \rho_s(0^+) \partial_u G_s(0^+) - \rho_s(0^-) \partial_u G_s(0^-) \} ds + \int_0^t \varphi_s(\rho, H) \delta G_s(0) ds. \end{aligned} \quad (2.13)$$

Remark 2.9. Any classical solution of (2.11) is actually a weak solution of (2.11). To verify it, suppose that ρ is a classical solution. Then, multiply both sides of the partial differential equation (2.11) by a test function G and integrate in time and space. Performing twice integration by parts and applying the boundary conditions leads to the integral equation (2.13).

We emphasize the fact we were not able to show uniqueness of weak solutions of (2.11) despite the effort of different techniques we have tried. The non-linearity in mixed boundary conditions of (2.11) lead to a very complicated problem of uniqueness. Sustaining our point of view that this is only a technical question, in Subsection A we prove uniqueness of strong solutions of (2.11).

Existence of weak solutions of (2.11) is a consequence of the tightness of the process, as we will see in Section 4. The assumption on uniqueness of weak solutions of (2.11) is also needed in the proof of large deviations, because its proof depends on the hydrodynamic limit for the WASEP with a slow bond.

Our first result is the hydrodynamic limit for the WASEP with a slow bond:

Theorem 2.10. *Suppose uniqueness of weak solutions of PDE (2.11). Let $H \in C^{1,2}([0, T] \times [0, 1])$. Fix a continuous initial profile $\gamma : \mathbb{T} \rightarrow [0, 1]$ and consider a sequence of probability measures μ_N on $\{0, 1\}^{\mathbb{T}_N}$ associated to γ in the sense (2.3). Then, for any $t \in [0, T]$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\mu_N}^H \left[\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \eta_t^H(x) - \int G(u) \rho_t(u) du \right| > \delta \right] = 0,$$

for every $\delta > 0$ and every function $G \in C(\mathbb{T})$, where ρ is the unique weak solution of (2.11) with $\rho_0 = \gamma$.

2.4. Large deviations principle

Denote by \mathcal{M}_0 the subset of \mathcal{M} of all absolutely continuous measures with density bounded by 1:

$$\mathcal{M}_0 = \left\{ \omega \in \mathcal{M} ; \omega(du) = \rho(u) du \quad \text{and} \quad 0 \leq \rho \leq 1 \quad \text{almost surely} \right\}.$$

The set \mathcal{M}_0 is a closed subset of \mathcal{M} endowed with the weak topology. This property is inherited by $\mathcal{D}([0, T], \mathcal{M}_0)$, which is a closed subset of $\mathcal{D}_{\mathcal{M}}$ for the Skorohod topology. We will denote $\mathcal{D}([0, T], \mathcal{M}_0)$ simply by $\mathcal{D}_{\mathcal{M}_0}$.

Definition 2.11. *Given $H \in C_k^{0,1}([0, T] \times (0, 1))$ define $\mathcal{E}_H : \mathcal{D}_{\mathcal{M}} \rightarrow \mathbb{R} \cup \{\infty\}$ by*

$$\mathcal{E}_H(\pi) = \begin{cases} \langle \partial_u H, \rho \rangle - 2 \langle H, H \rangle, & \text{if } \pi \in \mathcal{D}_{\mathcal{M}_0} \text{ and } \pi(du) = \rho(t, u) du, \\ \infty, & \text{otherwise.} \end{cases}$$

Furthermore, define the energy functional $\mathcal{E} : \mathcal{D}_{\mathcal{M}} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ by

$$\mathcal{E}(\pi) = \sup_H \mathcal{E}_H(\pi), \tag{2.14}$$

where the supremum is taken over functions $H \in C_k^{0,1}([0, T] \times (0, 1))$.

In Section 3.5 we prove that if $\pi \in \mathcal{D}_{\mathcal{M}}$ and $\mathcal{E}(\pi) < \infty$, then there exists $\rho \in L^2(0, T; \mathcal{H}^1(0, 1))$ such that $\pi(t, du) = \rho_t(u)du$. Keeping this in mind, given $H \in C^{1,2}([0, T] \times [0, 1])$ and $\pi \in \mathcal{D}_{\mathcal{M}}$, define

$$\hat{J}_H(\pi) = \ell_H(\pi) - \Phi_H(\pi), \quad (2.15)$$

where

$$\begin{aligned} \ell_H(\pi) &= \langle \rho_T, H_T \rangle - \langle \rho_0, H_0 \rangle - \int_0^T \langle \rho_t, (\partial_t + \Delta) H_t \rangle dt \\ &\quad - \int_0^T \{ \rho_t(0^+) \partial_u H_t(0^+) - \rho_t(0^-) \partial_u H_t(0^-) \} dt \\ &\quad + \int_0^T (\rho_t(0^+) - \rho_t(0^-)) \delta H_t(0) dt \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \Phi_H(\pi) &= \int_0^T \langle \chi(\rho_t), (\partial_u H_t)^2 \rangle dt + \int_0^T \rho_t(0^-) (1 - \rho_t(0^+)) \psi(\delta H_t(0)) dt \\ &\quad + \int_0^T \rho_t(0^+) (1 - \rho_t(0^-)) \psi(-\delta H_t(0)) dt, \end{aligned} \quad (2.17)$$

where $\psi(x) = e^x - x - 1$ and $\delta H_t(0) = H_t(0^+) - H_t(0^-)$. It is worth highlighting that, as functions of H , $\ell_H(\pi)$ is linear and $\Phi_H(\pi)$ is convex.

Definition 2.12. Given $H \in C^{1,2}([0, T] \times [0, 1])$, define the functional $J_H : \mathcal{D}_{\mathcal{M}} \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$J_H(\pi) = \begin{cases} \hat{J}_H(\pi), & \text{if } \mathcal{E}(\pi) < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 2.13. Let the rate functional $I : \mathcal{D}_{\mathcal{M}} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ be

$$I(\pi) = \sup_H J_H(\pi),$$

being the supremum above over functions $H \in C^{1,2}([0, T] \times [0, 1])$.

The large deviations study is decomposed in the study of deviations from the initial measure and deviations from the expected trajectory, see [KL, Chapter 10]. Since the large deviations for Bernoulli product measures are well known, we restrict ourselves to the deviations from the expected trajectory. We start henceforth the process from a sequence of *deterministic* initial configurations. This avoids the analysis of static large deviations, since we are interested here in dynamical large deviations. Recall that $\mathbb{Q}_{\mu_N}^N$ is the measure on the path space $\mathcal{D}_{\mathcal{M}}$ induced by the initial measure μ_N and the empirical process π_t^N introduced in (2.5). We are now in position to state the main result of the paper.

Theorem 2.14. *Let μ_N be a sequence of deterministic initial configurations associated to a bounded density profile $\gamma : \mathbb{T} \rightarrow \mathbb{R}$ in the sense of the Definition 2.2. Then, the sequence of measures $\{\mathbb{Q}_{\mu_N}^N; N \geq 1\}$ satisfies the following large deviation estimates:*

(i) **Upper bound:** *For any \mathcal{C} closed subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}^N [\mathcal{C}] \leq - \inf_{\pi \in \mathcal{C}} I(\pi).$$

(ii) **Lower bound for smooth profiles:** *For any \mathcal{O} open subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\underline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}^N [\mathcal{O}] \geq - \inf_{\pi \in \mathcal{O} \cap \mathcal{D}_{\mathcal{M}_0}^S} I(\pi),$$

where $\mathcal{D}_{\mathcal{M}_0}^S$ denotes the set of paths $\pi \in \mathcal{D}_{\mathcal{M}}$ such that $\pi_t(du) = \rho_t(u) du$ with $\rho \in C^{1,2}([0, T] \times [0, 1])$.

The item (i) of theorem above is proved in Section 5. The proof of item (ii) is presented in Section 6.

3. Superexponential replacement lemmas and energy estimate

Both in the proof of hydrodynamic limit for the WASEP with a slow bond and in the proof of the large deviations principle for the SSEP with a slow bond, replacement lemma and energy estimates play an important role. By a *replacement lemma* we mean a result that allows to replace the average time occupation in a site for the average time occupation in a box around that site. And by *energy estimates* we mean a result assuring that time trajectories of the empirical measure are asymptotically close to elements of a certain Sobolev space.

In the proof of large deviations we will need such results in a superexponential setting. In other words, the corresponding probabilities must converge to one in a faster way than exponentially.

3.1. Definitions and estimates lemmas

Denote by $\mathbf{H}(\mu_N | \nu_\alpha^N)$ the entropy of a probability measure μ_N with respect to the invariant measure ν_α^N . For a precise definition and properties of the entropy, see [KL]. It is well known the existence of a constant $K_0 := K_0(\alpha)$ such that

$$\mathbf{H}(\mu_N | \nu_\alpha^N) \leq K_0 N, \quad (3.1)$$

for any probability measure μ_N in Ω_N . See for instance the appendix of [FGN1].

Denote by $\langle \cdot, \cdot \rangle_{\nu_\alpha^N}$ the scalar product of $L^2(\nu_\alpha^N)$ and denote by \mathfrak{D}_N the Dirichlet form, which is the convex and lower semicontinuous functional (see [KL, Corollary A1.10.3]) defined by

$$\mathfrak{D}_N(f) = \langle -L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_\alpha^N},$$

where f is a probability density with respect to ν_α^N (i.e. $f \geq 0$ and $\int f d\nu_\alpha^N = 1$). An elementary computation shows that

$$\mathfrak{D}_N(f) = \sum_{x \in \mathbb{T}_N} \frac{\xi_x^N}{2} \int \left(\sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)} \right)^2 d\nu_\alpha^N(\eta),$$

where ξ_x^N is defined in (2.10).

From this point on, abusing of notation, we denote the biggest integer small or equal to εN simply by εN . Next, we define the local average of particles, which corresponds to the mean occupation in a box around a given site. The idea is to define a box around the site x in such a way it avoids the slow bond.

Definition 3.1. *If $x \in \mathbb{T}_N$ is such that $\frac{x}{N} \in \mathbb{T} \setminus (-\varepsilon, 0)$, we define the local average by*

$$\eta^{\varepsilon N}(x) = \frac{1}{\varepsilon N} \sum_{y=x+1}^{x+\varepsilon N} \eta(y).$$

If $\frac{x}{N} \in (-\varepsilon, 0)$, define the local average by

$$\eta^{\varepsilon N}(x) = \frac{1}{\varepsilon N} \sum_{y=-\varepsilon N}^{-1} \eta(y).$$

In accordance with to the previous definition of local density of particles, we define an approximation of identity ι_ε in the continuous torus by

$$\iota_\varepsilon(u, v) = \begin{cases} \frac{1}{\varepsilon} \mathbf{1}_{(v, v+\varepsilon)}(u), & \text{if } v \in \mathbb{T} \setminus (-\varepsilon, 0), \\ \frac{1}{\varepsilon} \mathbf{1}_{(-\varepsilon, 0)}(u), & \text{if } v \in (-\varepsilon, 0). \end{cases} \quad (3.2)$$

We also define the convolution

$$(\psi * \iota_\varepsilon)(v) = \langle \psi, \iota_\varepsilon(\cdot, v) \rangle,$$

for a function $\psi : \mathbb{T} \rightarrow \mathbb{R}$ or a measure ψ on the torus \mathbb{T} . The following identity is relevant:

$$(\pi^N * \iota_\varepsilon)\left(\frac{x}{N}\right) = \eta^{\varepsilon N}(x), \quad \text{for all } x \in \mathbb{T}_N. \quad (3.3)$$

To simplify notation, define the functions

$$g_1 : \{0, 1\}^{\mathbb{T}} \rightarrow \mathbb{R} \text{ by } g_1(\eta) = \eta(0)(1 - \eta(1)) \quad (3.4)$$

and

$$\tilde{g}_1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \text{ by } \tilde{g}_1(\alpha, \beta) = \alpha(1 - \beta).$$

Also,

$$g_2 : \{0, 1\}^{\mathbb{T}} \rightarrow \mathbb{R} \text{ by } g_2(\eta) = \eta(1)(1 - \eta(0)) \quad (3.5)$$

and

$$\tilde{g}_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \text{ by } \tilde{g}_2(\alpha, \beta) = \beta(1 - \alpha).$$

Lemma 3.2. Fix $F : \mathbb{T} \rightarrow \mathbb{R}$ and let f be a density with respect to ν_α^N . Then, for any $A > 0$ hold the inequalities

$$\begin{aligned} & \frac{1}{N} \sum_{x \neq -1} \int F\left(\frac{x}{N}\right) \left\{ \tau_x g_i(\eta) - \tilde{g}_i(\eta^{\varepsilon N}(x), \eta^{\varepsilon N}(x+1)) \right\} f(\eta) d\nu_\alpha^N(\eta) \\ & \leq 12A\varepsilon \sum_{x \neq -1} \left(F\left(\frac{x}{N}\right) \right)^2 + \frac{3}{A} \mathfrak{D}_N(f), \end{aligned} \quad (3.6)$$

$$\begin{aligned} & \frac{1}{N} \sum_{x \in \mathbb{T}_N} \int F\left(\frac{x}{N}\right) \{ \eta(x) - \eta^{\varepsilon N}(x) \} f(\eta) d\nu_\alpha^N(\eta) \\ & \leq 4A\varepsilon \sum_{x \in \mathbb{T}_N} \left(F\left(\frac{x}{N}\right) \right)^2 + \frac{1}{A} \mathfrak{D}_N(f), \end{aligned} \quad (3.7)$$

$$\begin{aligned} & F\left(\frac{-1}{N}\right) \int \left\{ \tau_{-1} g_i(\eta) - \tilde{g}_i(\eta^{\varepsilon N}(-1), \eta^{\varepsilon N}(0)) \right\} f(\eta) d\nu_\alpha^N(\eta) \\ & \leq 6A\varepsilon N \left(F\left(\frac{-1}{N}\right) \right)^2 + \frac{3}{A} \mathfrak{D}_N(f), \end{aligned} \quad (3.8)$$

$$\int \{ \eta(x) - \eta^{\varepsilon N}(x) \} f(\eta) d\nu_\alpha^N(\eta) \leq 4NA\varepsilon + \frac{1}{A} \mathfrak{D}_N(f), \quad \forall x \in \mathbb{T}_N, \quad (3.9)$$

with $i = 1, 2$.

Proof. The method of proof for the four inequalities is exactly the same. For this reason, we detail only the inequality (3.6) with $i = 1$. The reader can check the remaining inequalities. First, adding and subtracting terms, we rewrite $\tau_x g_1(\eta) - \tilde{g}_1(\eta^{\varepsilon N}(x), \eta^{\varepsilon N}(x+1))$ as

$$\eta(x) - \eta^{\varepsilon N}(x) - \eta(x)(\eta(x+1) - \eta^{\varepsilon N}(x+1)) - \eta^{\varepsilon N}(x+1)(\eta(x) - \eta^{\varepsilon N}(x)). \quad (3.10)$$

We handle the parcel $\eta(x)(\eta(x+1) - \eta^{\varepsilon N}(x+1))$ of above first. We claim that for f density with respect to ν_α^N and for any $A > 0$, it is true that

$$\begin{aligned} & \frac{1}{N} \sum_{x \neq -1} \int F\left(\frac{x}{N}\right) \eta(x) \left\{ \eta(x+1) - \eta^{\varepsilon N}(x+1) \right\} f(\eta) d\nu_\alpha^N(\eta) \\ & \leq 4A\varepsilon \sum_{x \neq -1} \left(F\left(\frac{x}{N}\right) \right)^2 + \frac{1}{A} \mathfrak{D}_N(f). \end{aligned} \quad (3.11)$$

Recall Definition 3.1. Let x be such that $\frac{x+1}{N} \notin (-\varepsilon, 0]$. In this case,

$$\begin{aligned} & \int F\left(\frac{x}{N}\right) \eta(x) (\eta(x+1) - \eta^{\varepsilon N}(x+1)) f(\eta) d\nu_{\alpha}^N(\eta) \\ &= \int F\left(\frac{x}{N}\right) \eta(x) \left\{ \frac{1}{\varepsilon N} \sum_{y=x+2}^{x+1+\varepsilon N} (\eta(x+1) - \eta(y)) \right\} f(\eta) d\nu_{\alpha}^N(\eta). \end{aligned}$$

Replacing $\eta(x+1) - \eta(y)$ by a telescopic sum, one can rewrite the expression above as

$$\int F\left(\frac{x}{N}\right) \eta(x) \left\{ \frac{1}{\varepsilon N} \sum_{y=x+2}^{x+1+\varepsilon N} \sum_{z=x+1}^{y-1} (\eta(z) - \eta(z+1)) \right\} f(\eta) d\nu_{\alpha}^N(\eta).$$

Rewriting the last expression as twice the half and making the change of variables $\eta \mapsto \eta^{z, z+1}$ (and using that the probability ν_{α}^N is invariant for this map) it becomes

$$\frac{1}{2\varepsilon N} \sum_{y=x+2}^{x+1+\varepsilon N} \sum_{z=x+1}^{y-1} F\left(\frac{x}{N}\right) \int \eta(x) (\eta(z) - \eta(z+1)) (f(\eta) - f(\eta^{z, z+1})) d\nu_{\alpha}^N(\eta).$$

By means of $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ and the Cauchy-Schwarz inequality, we bound the previous expression from above by

$$\begin{aligned} & \frac{1}{2\varepsilon N} \sum_{y=x+2}^{x+1+\varepsilon N} \sum_{z=x+1}^{y-1} \frac{A}{\xi_z^N} \left(F\left(\frac{x}{N}\right)\right)^2 \int \left(\sqrt{f(\eta)} + \sqrt{f(\eta^{z, z+1})} \right)^2 d\nu_{\alpha}^N(\eta) \\ &+ \frac{1}{2\varepsilon N} \sum_{y=x+2}^{x+1+\varepsilon N} \sum_{z=x+1}^{y-1} \frac{\xi_z^N}{A} \int \left(\sqrt{f(\eta)} - \sqrt{f(\eta^{z, z+1})} \right)^2 d\nu_{\alpha}^N(\eta), \end{aligned} \tag{3.12}$$

for any $A > 0$ and where ξ_z^N was defined in (2.10). The second sum above is bounded by

$$\frac{1}{A\varepsilon N} \sum_{y=x+2}^{x+1+\varepsilon N} \sum_{z \in \mathbb{T}_N} \frac{\xi_z^N}{2} \int \left(\sqrt{f(\eta)} - \sqrt{f(\eta^{z, z+1})} \right)^2 d\nu_{\alpha}^N(\eta) \leq \frac{1}{A} \mathfrak{D}_N(f).$$

Since $\xi_z^N = 1$ for all $z \in \{x+1, \dots, x+\varepsilon N\}$ and f is a density with respect to ν_{α}^N , the first term in (3.12) is bounded by

$$\frac{1}{\varepsilon N} \sum_{y=x+2}^{x+1+\varepsilon N} \sum_{z=x+1}^{y-1} 2A \left(F\left(\frac{x}{N}\right)\right)^2 \leq 2A\varepsilon N \left(F\left(\frac{x}{N}\right)\right)^2.$$

Therefore, for any site x such that $\frac{x+1}{N} \notin (-\varepsilon, 0]$,

$$\int F\left(\frac{x}{N}\right) \eta(x) (\eta(x+1) - \eta^{\varepsilon N}(x+1)) f(\eta) d\nu_{\alpha}^N(\eta) \leq 2A\varepsilon N \left(F\left(\frac{x}{N}\right)\right)^2 + \frac{1}{A} \mathfrak{D}_N(f).$$

Now, let x be a site such that $\frac{x+1}{N} \in (-\varepsilon, 0]$. In this case,

$$\begin{aligned} & \int F\left(\frac{x}{N}\right) \eta(x) (\eta(x+1) - \eta^{\varepsilon N}(x+1)) f(\eta) d\nu_{\alpha}^N(\eta) \\ &= \int F\left(\frac{x}{N}\right) \eta(x) \left\{ \frac{1}{\varepsilon N} \sum_{y=-\varepsilon N}^{-1} (\eta(x+1) - \eta(y)) \right\} f(\eta) d\nu_{\alpha}^N(\eta). \end{aligned}$$

We split the last sum into two blocks: $\{-1 - \varepsilon N + 1, \dots, x\}$ and $\{x+1, \dots, -1\}$. Then we proceed by writing $\eta(x+1) - \eta(y)$ as a telescopic sum, getting

$$\begin{aligned} & F\left(\frac{x}{N}\right) \int \eta(x) \left\{ \frac{1}{\varepsilon N} \sum_{y=-\varepsilon N}^x \sum_{z=y}^x (\eta(z+1) - \eta(z)) \right\} f(\eta) d\nu_{\alpha}^N(\eta) \\ &+ F\left(\frac{x}{N}\right) \int \eta(x) \left\{ \frac{1}{\varepsilon N} \sum_{y=x+2}^{-1} \sum_{z=x+1}^{y-1} (\eta(z) - \eta(z+1)) \right\} f(\eta) d\nu_{\alpha}^N(\eta). \end{aligned}$$

By the same arguments used above and since $\xi_z^N = 1$ for all z in the range $\{-\varepsilon N, \dots, -2\}$, we bound the previous expression by

$$4A\varepsilon N \left(F\left(\frac{x}{N}\right)\right)^2 + \frac{1}{A} \mathfrak{D}_N(f).$$

This proves (3.11). Analogous bounds for the remaining parcels in (3.10) lead to (3.6). \square

Lemma 3.3. *Fix any function $H : \mathbb{T} \rightarrow \mathbb{R}$ and let f be a density with respect to ν_{α}^N . Then,*

$$\begin{aligned} & \int \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \{\eta(x - \varepsilon N) - \eta(x)\} f(\eta) d\nu_{\alpha}^N(\eta) \\ & \leq N \mathfrak{D}_N(f) + \frac{2}{N} \sum_{x \in \mathbb{T}_N} \left(H\left(\frac{x}{N}\right)\right)^2 \left\{1 + \frac{1}{\varepsilon} \mathbf{1}_{(-\varepsilon, 0]}\left(\frac{x}{N}\right)\right\}. \end{aligned} \quad (3.13)$$

Moreover, this inequality remains valid replacing $\{\eta(x - \varepsilon N) - \eta(x)\}$ by $\{\eta(x) - \eta(x + \varepsilon N)\}$.

Proof. We begin by writing the left hand side of inequality (3.13) as a telescopic sum:

$$\begin{aligned} & \int \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \{\eta(x_0) - \eta(x_1)\} f(\eta) d\nu_{\alpha}^N(\eta) \\ &= \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \sum_{y=x_0}^{x_1-1} \int \{\eta(y) - \eta(y+1)\} f(\eta) d\nu_{\alpha}^N(\eta), \end{aligned}$$

where $x_0 = x - \varepsilon N$ and $x_1 = x$ (or $x_0 = x$ and $x_1 = x + \varepsilon N$ for the second case). Rewrite the expression above as twice the half. Then, making the changing of

variables $\eta \mapsto \eta^{x,x+1}$ on one piece and applying Young's inequality, we bound the previous expression by

$$\begin{aligned} & \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} \left(H\left(\frac{x}{N}\right)\right)^2 \sum_{y=x_0}^{x_1-1} \frac{A}{2\xi_y^N} \int \left\{ \sqrt{f(\eta)} + \sqrt{f(\eta^{y,y+1})} \right\}^2 d\nu_\alpha^N(\eta) \\ & + \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} \sum_{y=x_0}^{x_1-1} \frac{\xi_y^N}{2A} \int \left\{ \sqrt{f(\eta)} - \sqrt{f(\eta^{y,y+1})} \right\}^2 d\nu_\alpha^N(\eta), \quad \forall A > 0, \end{aligned} \quad (3.14)$$

where ξ_y^N was defined in (2.10). The second sum above is less than or equal to $\frac{1}{A} \mathfrak{D}_N(f)$. Since f is a density with respect to ν_α^N , the first sum in (3.14) is smaller or equal than

$$\frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} \left(H\left(\frac{x}{N}\right)\right)^2 \sum_{y=x_0}^{x_1-1} \frac{2A}{\xi_y^N} \leq \frac{2A}{\varepsilon N} \sum_{x \in \mathbb{T}_N} \left(H\left(\frac{x}{N}\right)\right)^2 \left\{ \varepsilon N + N \mathbf{1}_{(-\varepsilon, 0]} \left(\frac{x}{N}\right) \right\}.$$

This inequality is true for $x_0 = x - \varepsilon N$ and $x_1 = x$ or $x_0 = x$ and $x_1 = x + \varepsilon N$. Choosing $A = \frac{1}{N}$ completes the proof. \square

3.2. Superexponential replacement lemmas

In the large deviations proof, the replacement lemma presented in Section 3.11 is not enough, because we need to prove that the difference between cylinder functions and functions of the density field are superexponentially small, that is, of order smaller than $\exp\{-CN\}$, for any $C > 0$. We begin by exhibiting a superexponential replacement for the invariant measure ν_α^N .

Proposition 3.4. *Let $F_i: [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$, $i = 1, 2$, such that*

$$\overline{\lim}_{N \rightarrow \infty} \int_0^T \left((F_2(t, \frac{-1}{N}))^2 + \frac{1}{N} \sum_{x \neq -1} (F_1(t, \frac{x}{N}))^2 \right) dt < \infty.$$

For each $\varepsilon > 0$, consider

$$\begin{aligned} V_{N,\varepsilon}^{F_1, F_2}(t, \eta) &:= \frac{1}{N} \sum_{x \neq -1} F_1(t, \frac{x}{N}) \left\{ \tau_x g_1(\eta) - \tilde{g}_1(\eta^{\varepsilon N}(x), \eta^{\varepsilon N}(x+1)) \right\} \\ &+ F_2(t, \frac{-1}{N}) \left\{ \tau_{-1} g_1(\eta) - \tilde{g}_1(\eta^{\varepsilon N}(-1), \eta^{\varepsilon N}(0)) \right\}, \end{aligned}$$

where g_1 and \tilde{g}_1 have been defined in (3.4). Then, for any $\delta > 0$,

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\nu_\alpha^N} \left[\left| \int_0^T V_{N,\varepsilon}^{F_1, F_2}(t, \eta_t) dt \right| > \delta \right] = -\infty. \quad (3.15)$$

Finally, it is true the same result with g_2 and \tilde{g}_2 in lieu of g_1 and \tilde{g}_1 .

Proof. By

$$\overline{\lim}_N N^{-1} \log\{a_N + b_N\} = \max \left\{ \overline{\lim}_N N^{-1} \log a_N, \overline{\lim}_N N^{-1} \log b_N \right\}, \quad (3.16)$$

it is enough to prove (3.15) without the absolute value for $V_{N,\varepsilon}^{F_1, F_2}$ and $V_{N,\varepsilon}^{-F_1, -F_2}$. Let $C > 0$. By Chebyshev exponential inequality, we get

$$\begin{aligned} & \mathbb{P}_{\nu_\alpha^N} \left[\int_0^T V_{N,\varepsilon}^{F_1, F_2}(s, \eta_s) ds > \delta \right] \\ & \leq \exp\{-C\delta N\} \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ CN \int_0^T V_{N,\varepsilon}^{F_1, F_2}(s, \eta_s) ds \right\} \right]. \end{aligned}$$

To conclude the proof it is enough to assure that

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ \int_0^T CN V_{N,\varepsilon}^{F_1, F_2}(t, \eta_t) dt \right\} \right] \leq 0, \quad (3.17)$$

for every $C > 0$, because in this case we would have proved that left hand side of (3.15) is bounded from above by $-C\delta$ for an arbitrary $C > 0$. By the Feynman-Kac formula, for each fixed N the previous expectation is bounded from above by

$$\exp \left\{ \int_0^T \sup_f \left[\int CN V_{N,\varepsilon}^{F_1, F_2}(t, \eta) f(\eta) d\nu_\alpha^N(\eta) - N^2 \mathfrak{D}_N(f) \right] dt \right\},$$

where the supremum is carried over all density functions f with respect to ν_α^N . Replacing the expression of $V_{N,\varepsilon}^{F_1, F_2}(t, \eta)$ and using the Lemma 3.2 (notice that this lemma works for g_1 and g_2), we bound the expression in (3.17) by

$$\int_0^T \sup_f \left[6CA\varepsilon \left(2 \sum_{x \neq -1} \left(F_1(t, \frac{x}{N}) \right)^2 + N \left(F_2(t, \frac{-1}{N}) \right)^2 \right) + \frac{6C}{A} \mathfrak{D}_N(f) - N \mathfrak{D}_N(f) \right] dt.$$

Choosing $A = \frac{6C}{N}$, the expression above becomes

$$36C^2\varepsilon \int_0^T \left(\frac{2}{N} \sum_{x \neq -1} \left(F_1(t, \frac{x}{N}) \right)^2 + \left(F_2(t, \frac{-1}{N}) \right)^2 \right) dt,$$

which vanishes as $N \rightarrow \infty$ and then $\varepsilon \downarrow 0$. \square

Corollary 3.5. *Under the same hypothesis of the Proposition 3.4, for any $\delta > 0$,*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\left| \int_0^T V_{N,\varepsilon}^{F_1, F_2}(t, \eta_t) dt \right| > \delta \right] = -\infty. \quad (3.18)$$

Finally, the same result is still valid with g_2 and \tilde{g}_2 in lieu of g_1 and \tilde{g}_1 .

Proof. By the bound $\nu_\alpha^N(\eta) \geq (\alpha \wedge (1 - \alpha))^N$, we get

$$\begin{aligned}
 & \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\left| \int_0^T V_{N,\varepsilon}^{F_1, F_2}(t, \eta_t) dt \right| > \delta \right] \\
 &= \frac{1}{N} \log \left(\sum_{\eta \in \Omega_N} \mathbb{P}_\eta \left[\left| \int_0^T V_{N,\varepsilon}^{F_1, F_2}(t, \eta_t) dt \right| > \delta \right] \frac{\mu_N(\eta)}{\nu_\alpha^N(\eta)} \nu_\alpha^N(\eta) \right) \\
 &\leq \frac{1}{N} \log \left(\frac{1}{(\alpha \wedge (1 - \alpha))^N} \sum_{\eta \in \Omega_N} \mathbb{P}_\eta \left[\left| \int_0^T V_{N,\varepsilon}^{F_1, F_2}(t, \eta_t) dt \right| > \delta \right] \nu_\alpha^N(\eta) \right) \\
 &= \log \left(\frac{1}{(\alpha \wedge (1 - \alpha))^N} \right) + \frac{1}{N} \log \mathbb{P}_{\nu_\alpha^N} \left[\left| \int_0^T V_{N,\varepsilon}^{F_1, F_2}(t, \eta_t) dt \right| > \delta \right].
 \end{aligned}$$

Recalling Proposition 3.4 and $0 < \alpha < 1$, the limit (3.18) follows. \square

Corollary 3.6. *Given a bounded function $F : [0, T] \times \mathbb{T}$ and $x = -1$ or $x = 0$, let*

$$\hat{V}_{N,\varepsilon}^{F,x}(t, \eta) = F(t, \frac{x}{N}) \{ \eta(x) - \eta^{\varepsilon N}(x) \}.$$

Then, for any $\delta > 0$,

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\left| \int_0^T \hat{V}_{N,\varepsilon}^{F,x}(t, \eta_t) dt \right| > \delta \right] = -\infty. \quad (3.19)$$

Proof. We will prove the limit (3.19) for $\mu_N = \nu_\alpha^N$, to do this is enough to prove

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\nu_\alpha^N} \left[\int_0^T \hat{V}_{N,\varepsilon}^{G,x}(t, \eta_t) dt > \delta \right] = -\infty,$$

for $G = F$ and $G = -F$. This limit follows in the same sense as in the Proposition 3.4 and using (3.9) from Lemma 3.2. The extension for a general μ_N follows the same scheme in the proof of Corollary 3.5 and it is omitted here. \square

The next lemma is useful to get the results of the Subsection 3.4 from the results of this subsection.

Lemma 3.7. *If for any function $W_\varepsilon^N(t, \eta_t)$ uniformly bounded by C and for any $\delta > 0$ we have*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\left| \int_0^T W_\varepsilon^N(t, \eta_t) dt \right| > \delta \right] = -\infty,$$

then

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T W_\varepsilon^N(t, \eta_t) dt \right| \right] = 0.$$

Proof. Using that $W_\varepsilon^N(t, \eta_t)$ is uniformly bounded by C , the expectation $\mathbb{E}_{\mu_N} \left[\left| \int_0^T W_\varepsilon^N(t, \eta_t) dt \right| \right]$ is bounded from above by

$$\delta \mathbb{P}_{\mu_N} \left[\left| \int_0^T W_\varepsilon^N(t, \eta_t) dt \right| \leq \delta \right] + CT \mathbb{P}_{\mu_N} \left[\left| \int_0^T W_\varepsilon^N(t, \eta_t) dt \right| > \delta \right],$$

for any $\delta > 0$. Since for all $\delta > 0$ and $M > 0$, there exists ε_0 and N_0 such that

$$\mathbb{P}_{\mu_N} \left[\left| \int_0^T W_\varepsilon^N(t, \eta_t) dt \right| > \delta \right] \leq e^{-NM} < \delta/C, \quad \forall N \geq N_0 \text{ and } \forall \varepsilon < \varepsilon_0,$$

we have

$$\mathbb{E}_{\mu_N} \left[\left| \int_0^T W_\varepsilon^N(t, \eta_t) dt \right| \right] \leq 2\delta, \quad \forall N \geq N_0 \text{ and } \forall \varepsilon < \varepsilon_0,$$

which finishes the proof. \square

3.3. Supereponential energy estimate

Our goal here is to exclude trajectories with infinite energy in the large deviations regime. The next proposition is the key in the energy estimates.

Proposition 3.8. *Recall the Definition 2.11 of \mathcal{E}_H . For any function $H \in C_k^{0,1}([0, T] \times (0, 1))$, the following inequality holds:*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\mathcal{E}_H(\pi^N * \iota_\varepsilon) \geq l \right] \leq -l + K_0, \quad \forall l \in \mathbb{R}.$$

Proof. We begin by claiming that, for $\varepsilon > 0$ small enough,

$$\int_0^T \int_{\mathbb{T}} \partial_v H(t, v) (\pi_t^N * \iota_\varepsilon)(v) dv dt = \int_0^T \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H(t, \frac{x}{N}) [\eta_t(x) - \eta_t(x + \varepsilon N)] dt. \quad (3.20)$$

Since H has support contained in $[0, T] \times (\mathbb{T} \setminus \{0\})$ (using the identification of $(0, 1)$ with $\mathbb{T} \setminus \{0\}$), there exists some $\varepsilon_0 > 0$ such that $H(t, v)$ vanishes if $v \in (-\varepsilon_0, \varepsilon_0)$, for all $t \in [0, T]$. Applying Fubini's Theorem,

$$\int_0^T \int_{\mathbb{T}} \partial_u H(t, v) (\pi_t^N * \iota_\varepsilon)(v) dv dt = \int_0^T \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \left(\int_{\mathbb{T}} \partial_u H(t, v) \iota_\varepsilon(\frac{x}{N}, v) dv \right) dt.$$

From the definition of ι_ε given in (3.2) and taking $0 < \varepsilon < \varepsilon_0$, the last expression is equal to

$$\begin{aligned} & \int_0^T \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \left(\int_{\mathbb{T} \setminus (-\varepsilon, \varepsilon)} \partial_u H(t, v) \frac{1}{\varepsilon} \mathbf{1}_{(v, v+\varepsilon)}(\frac{x}{N}) dv \right) dt \\ &= \int_0^T \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \left(\frac{1}{\varepsilon} \mathbf{1}_{\mathbb{T} \setminus (-\varepsilon, \varepsilon)}(\frac{x}{N}) [H_t(\frac{x}{N}) - H_t(\frac{x}{N} - \varepsilon)] \right) dt. \end{aligned}$$

Using again that $H(t, v)$ vanishes if $v \in (-\varepsilon, \varepsilon)$, for all $t \in [0, T]$, the expression above is equal to

$$\int_0^T \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} \eta_t(x) [H_t(\frac{x}{N}) - H_t(\frac{x}{N} - \varepsilon)] dt,$$

proving the claim. Applying the definition of energy and (3.20), for $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} \mathcal{E}_H(\pi^N * \iota_\varepsilon) &= \int_0^T \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H(t, \frac{x}{N}) [\eta_t(x) - \eta_t(x + \varepsilon N)] dt \\ &\quad - 2 \int_0^T \int_{\mathbb{T}} (H(t, u))^2 du dt. \end{aligned}$$

Let us introduce the notation

$$V_N(\varepsilon, H, \eta) := \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H(\frac{x}{N}) \{\eta(x) - \eta(x + \varepsilon N)\} - \frac{2}{N} \sum_{x \in \mathbb{T}_N} (H(\frac{x}{N}))^2.$$

To achieve the statement of the proposition it is enough to have

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\int_0^T V_N(\varepsilon, H_t, \eta_t) dt \geq l \right] \leq -l + K_0.$$

By the Chebyshev exponential inequality,

$$\begin{aligned} &\frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\int_0^T V_N(\varepsilon, H_t, \eta_t) dt \geq l \right] \\ &\leq \frac{1}{N} \log \mathbb{E}_{\mu_N} \left[\exp \left\{ N \int_0^T V_N(\varepsilon, H_t, \eta_t) dt \right\} \right] - l. \end{aligned}$$

From Jensen's inequality, the entropy's inequality and the bound (3.1) of the relative entropy, the expectation in the right hand side of inequality above is bounded from above by

$$K_0 + \frac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ N \int_0^T V_N(\varepsilon, H_t, \eta_t) dt \right\} \right].$$

By the Feynman-Kac formula and the variational formula for the largest eigenvalue of a symmetric operator,

$$\begin{aligned} &\frac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ N \int_0^T V_N(\varepsilon, H_t, \eta_t) dt \right\} \right] \\ &\leq \int_0^T \sup_f \left\{ \int V_N(\varepsilon, H_t, \eta) f(\eta) d\nu_\alpha^N(\eta) - N \mathfrak{D}_N(f) \right\} dt, \end{aligned}$$

being the supremum above taken over all probability densities f with respect to ν_α^N . Recalling Lemma 3.3, we bound the last expression by

$$\int_0^T \frac{2}{\varepsilon N} \sum_{\frac{x}{N} \in (-\varepsilon, 0]} \left(H_t\left(\frac{x}{N}\right) \right)^2 dt.$$

Since H has compact support, for $\varepsilon > 0$ small enough the expression above vanishes. \square

Corollary 3.9. *For any functions $H_1, \dots, H_k \in C_k^{0,1}([0, T] \times (0, 1))$ holds*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi^N * \iota_\varepsilon) \geq l \right] \leq -l + K_0. \quad (3.21)$$

Proof. Straightforward from Proposition 3.8 and inequalities (3.16) and

$$\exp \left\{ \max_{1 \leq j \leq k} a_j \right\} \leq \sum_{1 \leq j \leq k} \exp \{a_j\}. \quad (3.22)$$

\square

We present the following lemma to get the results of the Subsection 3.5 from the results of this subsection.

Lemma 3.10. *If for any function $W_\varepsilon^N(\eta.)$ uniformly bounded by C and for any $\ell \in \mathbb{R}$, we have*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[W_\varepsilon^N(\eta.) \geq \ell \right] \leq -\ell + K_0,$$

then

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[W_\varepsilon^N(\eta.) \right] \leq K_0.$$

Proof. As in Lemma 3.7, the expectation $\mathbb{E}_{\mu_N} \left[W_\varepsilon^N(\eta.) \right]$ is bounded from above by

$$\ell + CT \mathbb{P}_{\mu_N} \left[W_\varepsilon^N(\eta.) \geq \ell \right],$$

for any $\ell \in \mathbb{R}$. Let $\delta > 0$. Take $\ell = K_0 + \delta$, then there exists ε_0 and N_0 such that

$$\mathbb{P}_{\mu_N} \left[W_\varepsilon^N(\eta.) \geq \ell \right] \leq e^{-N\delta}, \quad \forall N \geq N_0 \text{ and } \forall \varepsilon < \varepsilon_0,$$

we have

$$\mathbb{E}_{\mu_N} \left[W_\varepsilon^N(\eta.) \right] \leq \ell + e^{-N\delta}, \quad \forall N \geq N_0 \text{ and } \forall \varepsilon < \varepsilon_0.$$

Thus

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[W_\varepsilon^N(\eta.) \right] \leq \ell = K_0 + \delta,$$

for all $\delta > 0$, which finishes the proof. \square

3.4. Replacement Lemma

Proposition 3.11 (Replacement Lemma). *Let $F : \mathbb{T} \rightarrow \mathbb{R}$ be a bounded function and $(\mu_N)_{N \geq 1}$ any sequence of measures. Then, with $i = 1, 2$,*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t \frac{1}{N} \sum_{x \neq -1} F\left(\frac{x}{N}\right) \left\{ \tau_x g_i(\eta_s) - \tilde{g}_i(\eta_s^{\varepsilon N}(x), \eta_s^{\varepsilon N}(x+1)) \right\} ds \right| \right] = 0, \quad (3.23)$$

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} F\left(\frac{x}{N}\right) \{ \eta_s(x) - \eta_s^{\varepsilon N}(x) \} ds \right| \right] = 0,$$

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^t \left\{ \tau_{-1} g_i(\eta_s) - \tilde{g}_i(\eta_s^{\varepsilon N}(-1), \eta_s^{\varepsilon N}(0)) \right\} ds \right| \right] = 0, \quad \forall t \in [0, T],$$

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\left| \int_0^T \{ \eta_s(x) - \eta_s^{\varepsilon N}(x) \} ds \right| \right] = 0, \quad \text{for } x = -1, 0.$$

This proposition can be obtained as a consequence of the Lemma 3.7 and the Corollaries 3.5 and 3.6, but just for the sake of completeness we present here an alternative proof.

Proof. We detail the proof of the first limit, being the others similar. By the definition of the entropy and Jensen's inequality, the expectation in (3.23) is bounded from above by

$$\frac{H(\mu_N | \nu_\alpha^N)}{\gamma N} + \frac{1}{\gamma N} \log \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ \gamma \left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} F\left(\frac{x}{N}\right) \{ \eta_s(x) - \eta_s^{\varepsilon N}(x) \} ds \right| \right\} \right],$$

for all $\gamma > 0$. In view of (3.1), it is enough to show that the second term vanishes as $N \rightarrow \infty$ and then $\varepsilon \downarrow 0$ for every $\gamma > 0$. Since $e^{|x|} \leq e^x + e^{-x}$, we may remove the absolute value inside the exponential. Thus, to complete the prove of this proposition, we need to show that

$$\frac{1}{\gamma N} \log \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ \gamma \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} F\left(\frac{x}{N}\right) \{ \eta_s(x) - \eta_s^{\varepsilon N}(x) \} ds \right\} \right]$$

goes to zero when $N \rightarrow \infty$ and then $\varepsilon \downarrow 0$. By the Feynman-Kac formula¹, for each fixed N the previous expression is bounded from above by

$$t \sup_f \left\{ \int \frac{1}{N} \sum_{x \in \mathbb{T}_N} F\left(\frac{x}{N}\right) \{ \eta(x) - \eta^{\varepsilon N}(x) \} f(\eta) d\nu_\alpha^N(\eta) - \frac{N}{\gamma} \mathfrak{D}_N(f) \right\},$$

where the supremum is carried over all density functions f with respect to ν_α^N . From inequality (3.7) of the Lemma 3.2, the previous expression is less than or equal to

$$t \sup_f \left\{ 4A\varepsilon \sum_{x \in \mathbb{T}_N} \left(F\left(\frac{x}{N}\right) \right)^2 + \frac{1}{A} \mathfrak{D}_N(f) - \frac{N}{\gamma} \mathfrak{D}_N(f) \right\}.$$

¹c.f. [KL, Lemma 7.2, page 336]

Choosing $A = \frac{\gamma}{N}$, last expression becomes

$$\frac{4\gamma\varepsilon t}{N} \sum_{x \in \mathbb{T}_N} \left(F\left(\frac{x}{N}\right)\right)^2,$$

which vanishes as $N \rightarrow \infty$ and then $\varepsilon \downarrow 0$, concluding the proof of first limit in statement of the lemma. \square

3.5. Sobolev space

We prove in this section that any limit point \mathbb{Q}^* of the sequence $\mathbb{Q}_{\mu_N}^N$ is concentrated on trajectories $\rho(t, u)du$ which belongs to a certain Sobolev space to be defined ahead. Let \mathbb{Q}^* be a limit point of the sequence $\mathbb{Q}_{\mu_N}^N$ along some subsequence.

Proposition 3.12. *The measure \mathbb{Q}^* is concentrated on paths $\rho_t(u)du$ such that $\rho \in L^2(0, T; \mathcal{H}^1(0, 1))$.*

The proof is based on the Riesz Representation Theorem and follows from the next lemma.

Lemma 3.13.

$$E_{\mathbb{Q}^*} \left[\sup_H \left\{ \int_0^T \int_{\mathbb{T}} \partial_u H(s, u) \rho_s(u) du ds - 2 \int_0^T \int_{\mathbb{T}} H(s, u)^2 du ds \right\} \right] \leq K_0,$$

where the supremum is carried over all functions H in $C_k^{0,1}([0, T] \times (0, 1))$.

There are two ways to prove this lemma, the classical one is a consequence of several lemmas, which we present after the proof of Proposition 3.12, following the ideas of [FL]. The other one is just a consequence of the Corollary 3.9 and the Lemma 3.10.

Proof of Proposition 3.12. Denote by $\ell : C_k^{0,1}([0, T] \times (0, 1)) \rightarrow \mathbb{R}$ the linear functional defined by

$$\ell(H) = \int_0^T \int_{\mathbb{T}} \partial_u H(s, u) \rho_s(u) du ds.$$

Since $C_k^{0,1}([0, T] \times (0, 1))$ is dense in $L^2([0, T] \times \mathbb{T})$, by Lemma 3.13, \mathbb{Q}^* -almost surely ℓ is a bounded linear functional on $C_k^{0,1}([0, T] \times (0, 1))$. Therefore we can extend ℓ to a \mathbb{Q}^* -almost surely bounded functional in $L^2([0, T] \times \mathbb{T})$. By the Riesz Representation Theorem, there exists a function G in $L^2([0, T] \times \mathbb{T})$ such that

$$\ell(H) = - \int_0^T \int_{\mathbb{T}} H(s, u) G(s, u) du ds,$$

concluding the proof. \square

For a function $H : \mathbb{T} \rightarrow \mathbb{R}$, $\varepsilon > 0$ and a positive integer N , define $U_N(\varepsilon, H, \eta)$ by

$$\begin{aligned} U_N(\varepsilon, H, \eta) &= \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \left\{ \eta(x - \varepsilon N) - \eta(x) \right\} \\ &\quad - \frac{2}{N} \sum_{x \in \mathbb{T}_N} \left(H\left(\frac{x}{N}\right) \right)^2 \left\{ 1 + \frac{1}{\varepsilon} \mathbf{1}_{(-\varepsilon, 0]}\left(\frac{x}{N}\right) \right\}. \end{aligned} \quad (3.24)$$

Recall the definition of the constant K_0 given in (3.1).

Lemma 3.14. *For $k \geq 1$, let $H_1, \dots, H_k : [0, T] \times \mathbb{T} \rightarrow \mathbb{R}$ be bounded functions. Then, for every $\varepsilon > 0$,*

$$\overline{\lim}_{\delta \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\max_{1 \leq i \leq k} \left\{ \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s^{\delta N}) ds \right\} \right] \leq K_0.$$

Proof. By Proposition 3.11, in order to prove this lemma it is sufficient to show that

$$\overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N} \left[\max_{1 \leq i \leq k} \left\{ \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s) ds \right\} \right] \leq K_0.$$

By the definition of the entropy and Jensen's inequality, the previous expectation is bounded from above by

$$\frac{H(\mu_N | \nu_\alpha^N)}{N} + \frac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ \max_{1 \leq i \leq k} \left\{ N \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s) ds \right\} \right\} \right].$$

By (3.1), the first parcel above is smaller than K_0 . By (3.22) and (3.16), the limit as $N \rightarrow \infty$ of the previous expression is bounded from above by

$$K_0 + \max_{1 \leq i \leq k} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}_{\nu_\alpha^N} \left[\exp \left\{ N \int_0^T U_N(\varepsilon, H_i(s, \cdot), \eta_s) ds \right\} \right].$$

We claim that the the lim sup above is non positive for each fixed i (and therefore the maximum in $i = 1, \dots, k$). Fix $1 \leq i \leq k$. By Feynman-Kac's formula² and the variational formula for the largest eigenvalue of a symmetric operator, for each fixed N , the second term in the previous expression is bounded from above by

$$\int_0^T \sup_f \left\{ \int U_N(\varepsilon, H_i(s, \cdot), \eta_s) f(\eta) d\nu_\alpha^N(\eta) - N \mathfrak{D}_N(f) \right\} ds.$$

In last formula the supremum is taken over all probability densities f with respect to ν_α^N . Applying Lemma 3.3 finishes the proof. \square

Proof of the Lemma 3.13. Consider a sequence $\{H_\ell, \ell \geq 1\}$ dense in $C_k^{0,1}([0, T] \times (0, 1))$ with respect to the norm $\|H\|_\infty + \|\partial_u H\|_\infty$. Recall that we suppose that

²See [KL], Lemma 7.2, p. 336.

$\mathbb{Q}_{\mu_N}^N$ converges to \mathbb{Q}^* . By Lemma 3.14, for every $k \geq 1$,

$$\begin{aligned} \overline{\lim}_{\delta \downarrow 0} E_{\mathbb{Q}^*} \left[\max_{1 \leq i \leq k} \left\{ \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}} H_i(s, u) [\rho_s^\delta(u - \varepsilon) - \rho_s^\delta(u)] du ds \right. \right. \\ \left. \left. - 2 \int_0^T \int_{\mathbb{T}} (H_i(s, u))^2 \{1 + \frac{1}{\varepsilon} \mathbf{1}_{(-\varepsilon, 0]}(u)\} du ds \right\} \right] \leq K_0, \end{aligned}$$

where $\rho^\delta(u) := (\rho * \iota_\delta)(u)$. Letting $\delta \downarrow 0$, we obtain

$$\begin{aligned} E_{\mathbb{Q}^*} \left[\max_{1 \leq i \leq k} \left\{ \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}} H_i(s, u) [\rho_s(u - \varepsilon) - \rho_s(u)] du ds \right. \right. \\ \left. \left. - 2 \int_0^T \int_{\mathbb{T}} (H_i(s, u))^2 \{1 + \frac{1}{\varepsilon} \mathbf{1}_{(-\varepsilon, 0]}(u)\} du ds \right\} \right] \leq K_0. \end{aligned}$$

Changing variables in the first integral,

$$\begin{aligned} E_{\mathbb{Q}^*} \left[\max_{1 \leq i \leq k} \left\{ \int_0^T \int_{\mathbb{T}} \frac{1}{\varepsilon} [H_i(s, u + \varepsilon) - H_i(s, u)] \rho_s(u) du ds \right. \right. \\ \left. \left. - 2 \int_0^T \int_{\mathbb{T}} (H_i(s, u))^2 \{1 + \frac{1}{\varepsilon} \mathbf{1}_{(-\varepsilon, 0]}(u)\} du ds \right\} \right] \leq K_0. \end{aligned}$$

Since $H_i \in C_k^{0,1}([0, T] \times (0, 1))$, this function vanishes in a neighborhood of zero. Making $\varepsilon \downarrow 0$ in the last inequality, we obtain

$$E_{\mathbb{Q}^*} \left[\max_{1 \leq i \leq k} \left\{ \int_0^T \int_{\mathbb{T}} \partial_u H_i(s, u) \rho_s(u) du ds - 2 \int_0^T \int_{\mathbb{T}} (H_i(s, u))^2 du ds \right\} \right] \leq K_0.$$

To conclude it remains to apply the Monotone Convergence Theorem and recall that $\{H_\ell, \ell \geq 1\}$ is a dense sequence. □

4. Hydrodynamic limit of the WASEP with a slow bond

Fix a function $H \in C^{1,2}([0, T] \times [0, 1])$. The probability $\mathbb{P}_{\mu_N}^H$ corresponds to the non-homogeneous Markov process $\eta_t = \eta_t^H$ with generator L_N^H defined in (2.9) accelerated by N^2 and with initial measure μ_N . We remark that μ_N is not invariant. Denote by $\mathbb{Q}_{\mu_N}^H$ the probability measure on the space of trajectories $\mathcal{D}_{\mathcal{M}}$ induced by the empirical measure π_t^N .

Proposition 4.1. *Consider a bounded density profile $\rho_0 : \mathbb{T} \rightarrow \mathbb{R}$ and $H \in C^{1,2}([0, T] \times [0, 1])$. The sequence of probabilities $\{\mathbb{Q}_{\mu_N}^H; N \geq 1\}$ converges in distribution to the probability measure concentrated on the absolutely continuous path $\pi_t(du) = \rho_t(u)du$, where density $\rho_t(u)$ is the unique weak solution of the partial differential equation (2.11).*

Observe that the Theorem 2.10 is a corollary of the previous proposition. The proof of above is divided in two parts. In Subsection 4.1, we show that the sequence $\{\mathbb{Q}_{\mu_N}^H; N \geq 1\}$ is tight. Subsection 4.4 is reserved to the characterization of limit points of the sequence.

Uniqueness of limit points is assumed, since we were not able to prove uniqueness of weak solutions of the partial differential equation (2.11). Additionally, uniqueness of strong solutions of (2.11) is presented in Appendix A.

4.1. Tightness

In this subsection we present the tightness of $\{\mathbb{Q}_{\mu_N}^H\}$.

Proposition 4.2. *For fixed $H \in C^{1,2}([0, T] \times [0, 1])$, the sequence of measures $\{\mathbb{Q}_{\mu_N}^H; N \geq 1\}$ is tight in the Skorohod topology of $\mathcal{D}_{\mathcal{M}}$.*

Proof. In order to prove tightness of the sequence of measures $\{\mathbb{Q}_{\mu_N}^H; N \geq 1\}$ induced in the Skorohod space $\mathcal{D}_{\mathcal{M}}$ by the random elements $\{\pi_t^N; 0 \leq t \leq T\}$ it is sufficient to prove that the sequence of stochastic processes $\langle \pi_t^N, H \rangle$ is tight. We begin by considering the martingale

$$M_{N,t}^H(G) = \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, \partial_s G_s \rangle + N^2 L_{N,s}^H \langle \pi_s^N, G_s \rangle ds, \quad (4.1)$$

with $H, G \in C^{1,2}([0, T] \times [0, 1])$. To prove tightness would be enough to handle the martingale above in the case $G \in C^2(\mathbb{T})$. However, for future applications in the characterization of limit points, we treat here the slightly more general setting $G \in C^{1,2}([0, T] \times [0, 1])$.

First, let us show that the $L^2(\mathbb{P}_{\mu_N}^H)$ -norm of this martingale vanishes as $N \rightarrow \infty$. The quadratic variation of $M_{N,t}^H(G)$ is given by

$$\langle M_N^H(G) \rangle_t = \int_0^t N^2 \left[L_{N,s}^H \langle \pi_s^N, G_s \rangle^2 - 2 \langle \pi_s^N, G_s \rangle L_{N,s}^H \langle \pi_s^N, G_s \rangle \right] ds.$$

Applying definition (2.9), the quadratic variation $\langle M_N^H(G) \rangle_t$ can be rewritten as

$$\int_0^t \sum_{x \in \mathbb{T}_N} \xi_x^N (\delta_N G_x)^2 \left[e^{\delta_N H_x \eta_s(x)(1-\eta_s(x+1))} + e^{-\delta_N H_x \eta_s(x+1)(1-\eta_s(x))} \right] ds$$

where $\delta_N F_x$ denotes $F_s(\frac{x+1}{N}) - F_s(\frac{x}{N})$ and $\xi_x^N = N^{-1}$ if $x = -1$, and $\xi_x^N = 1$ otherwise. We observe that $\delta_N F_x$ depends on s , but the dependence is dropped by convenience of notation. Since $H, G \in C^{1,2}([0, T] \times [0, 1])$, the expression above for the quadratic variation can be easily bounded by N^{-1} times a constant not depending on N . By Doob inequality, we conclude that the supremum norm of the martingale $M_{N,t}^H(G)$ goes to zero in probability as N goes to infinity. Hence $\{M_{N,t}^H(G)\}_{N \in \mathbb{N}}$ is tight.

Expression $\langle \pi_0^N, G_0 \rangle$ is bounded and constant in time, thus tight as well. It remains to analyze the tightness of the integral term in (4.1). Using Taylor's expansion in the exponentials and performing some elementary computations, expression $N^2 L_{N,s}^H \langle \pi_s^N, G_s \rangle$ can be written in the form

$$\begin{aligned}
 & N \sum_{x \neq -1, 0} \eta_s(x) \left[G_s\left(\frac{x+1}{N}\right) + G_s\left(\frac{x-1}{N}\right) - 2G_s\left(\frac{x}{N}\right) \right] \\
 & + N \sum_{x \neq -1} \left[\eta_s(x)(1 - \eta_s(x+1)) + \eta_s(x+1)(1 - \eta_s(x)) \right] (\delta_N H_x) (\delta_N G_x) \\
 & + \left[\eta_s(-1)(1 - \eta_s(0)) e^{\delta_N H_{-1}} - \eta_s(0)(1 - \eta_s(-1)) e^{-\delta_N H_{-1}} \right] \delta_N G_{-1} \\
 & + N \left[\eta_s(0) \delta_N G_0 - \eta_s(-1) \delta_N G_{-2} \right] + O_{H,G}\left(\frac{1}{N}\right).
 \end{aligned} \tag{4.2}$$

Again by smoothness of H and G , the expression above is uniformly bounded in time. Hence, this integral term in (4.1) is uniformly continuous. By Arzelà-Ascoli, the integral term is relatively compact, therefore tight. Since a finite sum of tight stochastic processes is tight, the proof is finished. \square

4.2. Radon-Nikodym derivative

In this section we deal with the Radon-Nikodym derivative between the SSEP with a slow bond and the WASEP with a slow bond. Its formula will be useful both in the proof of the hydrodynamic limit for the WASEP with a slow bond and in the proof of the large deviations for the SSEP with a slow bond.

By $(\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N})(t)$ we denote the Radon-Nikodym derivative of $\mathbb{P}_{\mu_N}^H$ with respect to \mathbb{P}_{μ_N} restricted to the σ -algebra generated by $\{\eta_s, 0 \leq s \leq t\}$. It is a general fact of stochastic processes that $(\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N})(t)$ is a mean-one positive martingale. The explicit formula of the Radon-Nikodym derivative between two Markov process on a countable space state³ shows that $(\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N})(T)$ is equal to

$$\exp \left\{ N \left[\langle \pi_T^N, H_T \rangle - \langle \pi_0^N, H_0 \rangle - \frac{1}{N} \int_0^T e^{-N \langle \pi_t^N, H_t \rangle} (\partial_t + N^2 L_N) e^{N \langle \pi_t^N, H_t \rangle} dt \right] \right\}. \tag{4.3}$$

We are going to write just $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ for $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}(T)$, since the time horizon $T > 0$ is fixed. Recall the notation $\delta_N H_x = H_t(\frac{x+1}{N}) - H_t(\frac{x}{N})$. Performing

³See Appendix of [KL]

elementary calculations, we can rewrite (4.3) as

$$\begin{aligned} & \exp \left\{ N \langle \pi_T^N, H_T \rangle - N \langle \pi_0^N, H_0 \rangle - N \int_0^T \langle \pi_t^N, \partial_t H_t \rangle dt \right. \\ & - N^2 \int_0^T \sum_{x \in \mathbb{T}_N} \xi_x^N \eta_t(x) (1 - \eta_t(x+1)) (e^{\delta_N H_x} - 1) dt \\ & \left. - N^2 \int_0^T \sum_{x \in \mathbb{T}_N} \xi_x^N \eta_t(x+1) (1 - \eta_t(x)) (e^{-\delta_N H_x} - 1) dt \right\}. \end{aligned} \quad (4.4)$$

Since $H \in C^{1,2}([0, T] \times [0, 1])$, by Taylor's expansion and the inequality $|e^u - 1 - u - (1/2)u^2| \leq (1/6)|u|^3 e^{|u|}$, we observe that all the expressions

- $\frac{1}{N} \sum_{x \neq -1, 0} \eta_t(x) N^2 (\delta_N H_x - \delta_N H_{x-1}) - \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \partial_u^2 H_t(\frac{x}{N})$,
- $N^2 (e^{\pm \delta_N H_x} \mp \delta_N H_x - 1) - \frac{1}{2} (\partial_u H_t)^2(\frac{x}{N})$
- $N \delta_N H_0 - \partial_u H_t(\frac{0}{N})$ and $N \delta_N H_{-2} - \partial_u H_t(\frac{-1}{N})$

are, in modulus, of order $\frac{1}{N}$. Putting together the facts above, we can rewrite (4.4) as

$$\begin{aligned} & \exp \left\{ N \left[\langle \pi_T^N, H_T \rangle - \langle \pi_0^N, H_0 \rangle - \int_0^T \langle \pi_t^N, (\partial_t + \Delta) H_t \rangle dt \right. \right. \\ & - \int_0^T \left\{ \eta_t(0) \partial_u H_t(\frac{0}{N}) - \eta_t(-1) \partial_u H_t(\frac{-1}{N}) \right\} dt + O_{H,T}(\frac{1}{N}) \\ & - \int_0^T \frac{1}{N} \sum_{x \neq -1} \left[\eta_t(x) (1 - \eta_t(x+1)) + \eta_t(x+1) (1 - \eta_t(x)) \right] \frac{1}{2} (\partial_u H_t)^2(\frac{x}{N}) dt \\ & \left. \left. - \int_0^T \eta_t(-1) (1 - \eta_t(0)) (e^{\delta_N H_{-1}} - 1) dt - \int_0^T \eta_t(0) (1 - \eta_t(-1)) (e^{-\delta_N H_{-1}} - 1) dt \right] \right\}. \end{aligned} \quad (4.5)$$

As we shall see, the expression above is enough in order to prove the hydrodynamical limit of the WASEP with a slow bond. Further estimates on the Radon-Nikodym derivative will be presented at Section 5.

4.3. Sobolev space

In this section, we prove that any limit point \mathbb{Q}_*^H of the sequence $\mathbb{Q}_{\mu_N}^H$ is concentrated on trajectories $\rho_t(u) du$ belonging the Sobolev space of Definition 2.4. By expression (4.5), there exists a constant $C(H, T) > 0$ not depending on N such that

$$\left\| \frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}} \right\|_{\infty} \leq \exp \{ C(H, T) N \}. \quad (4.6)$$

Proposition 4.3. *Given a bounded function $G : \mathbb{T} \rightarrow \mathbb{R}$, then, for all $t \in [0, T]$,*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^H \left[\left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \{\eta_s(x) - \eta_s^{\varepsilon N}(x)\} ds \right| \right] = 0,$$

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^H \left[\left| \int_0^t \frac{1}{N} \sum_{x \neq -1} G\left(\frac{x}{N}\right) \left\{ \tau_x g_i(\eta_s) - \tilde{g}_i(\eta_s^{\varepsilon N}(x), \eta_s^{\varepsilon N}(x+1)) \right\} ds \right| \right] = 0,$$

and

$$\overline{\lim}_{\varepsilon \rightarrow 0} \overline{\lim}_{N \rightarrow \infty} \mathbb{E}_{\mu_N}^H \left[\left| \int_0^t G\left(\frac{-1}{N}\right) \left\{ \tau_{-1} g_i(\eta_s) - \tilde{g}_i(\eta_s^{\varepsilon N}(-1), \eta_s^{\varepsilon N}(0)) \right\} ds \right| \right] = 0,$$

where g_i and \tilde{g}_i with $i = 1, 2$ have been defined in (3.4) and (3.5).

Proof. Let us prove the first limit above. Fix $\gamma > 0$. From definition of the entropy and Jensen's inequality, the expectation appearing there is bounded from above by

$$\frac{H(\mu_N | \nu_\alpha^N)}{\gamma N} + \frac{1}{\gamma N} \log \mathbb{E}_{\nu_\alpha^N}^H \left[\exp \left\{ \gamma \left| \int_0^t \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \{\eta_s(x) - \eta_s^{\varepsilon N}(x)\} ds \right| \right\} \right].$$

In view of (3.1), it is enough to show that the second term vanishes as $N \rightarrow \infty$ and then $\varepsilon \downarrow 0$ for every $\gamma > 0$. By (4.6), the expression above is bounded by

$$\frac{H(\mu_N | \nu_\alpha^N)}{\gamma N} + \frac{C(H, T)}{\gamma} + \frac{1}{\gamma N} \log \mathbb{E}_{\nu_\alpha^N}^H \left[\exp \left\{ \gamma \left| \int_0^t \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \{\eta_s(x) - \eta_s^{\varepsilon N}(x)\} ds \right| \right\} \right].$$

Invoking proof of the Proposition 3.11 and noticing that γ is arbitrary large gives the result. The remaining limits follow analogous steps. \square

Proposition 4.4. *The measure \mathbb{Q}_*^H is concentrated on paths $\rho_t(u)du$ such that $\rho \in L^2(0, T; \mathcal{H}^1(0, 1))$.*

Proof. As before, the proof of this result follows from Proposition 3.12 put together with estimate (4.6). \square

4.4. Characterization of limit points

Here, we prove that all limit points of the sequence $\{\mathbb{Q}_{\mu_N}^H : N \geq 1\}$ are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure: $\pi(t, du) = \rho_t(u)du$, whose density $\rho_t(u)$ is a weak solution of the hydrodynamic equation (2.11).

Let \mathbb{Q}_*^H be a limit point of the sequence $\{\mathbb{Q}_{\mu_N}^H : N \geq 1\}$ and assume, without loss of generality, that $\{\mathbb{Q}_{\mu_N}^H : N \geq 1\}$ converges to \mathbb{Q}_*^H . The existence of \mathbb{Q}_*^H is guaranteed by Proposition 4.2.

In Proposition 4.4, we have proved that $\rho_t(\cdot)$ belongs to $L^2(0, T; \mathcal{H}^1(0, 1))$. It is well known that the Sobolev space $\mathcal{H}^1(0, 1)$ has special properties: its elements are absolutely continuous functions with bounded variation, c.f. [Evans], with well-defined side limits at zero. Such property is inherited by $L^2(0, T; \mathcal{H}^1(0, 1))$ in the sense that we can integrate in time the side limits at the boundaries. Let $G \in C^{1,2}([0, T] \times [0, 1])$. We begin by claiming that

$$\begin{aligned} \mathbb{Q}_*^H \left[\pi. : \langle \rho_t, G_t \rangle - \langle \rho_0, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta) G_s \rangle ds \right. \\ \left. - 2 \int_0^t \langle \chi(\rho_s), \partial_u H_s \partial_u G_s \rangle ds - \int_0^t \varphi_s(\rho, H) \delta G_s(0) ds \right. \\ \left. - \int_0^t \{ \rho_s(0^+) \partial_u G_s(0^+) - \rho_s(0^-) \partial_u G_s(0^-) \} ds = 0, \forall t \in [0, T] \right] = 1, \end{aligned} \quad (4.7)$$

where $\varphi_s(\rho, H)$ was defined in (2.12). In order to prove the equality above, it is enough to show

$$\begin{aligned} \mathbb{Q}_*^H \left[\pi. : \sup_{0 \leq t \leq T} \left| \langle \rho_t, G_t \rangle - \langle \rho_0, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta) G_s \rangle ds \right. \right. \\ \left. - 2 \int_0^t \langle \chi(\rho_s), \partial_u H_s \partial_u G_s \rangle ds - \int_0^t \varphi_s(\rho, H) \delta G_s(0) ds \right. \\ \left. - \int_0^t \{ \rho_s(0^+) \partial_u G_s(0^+) - \rho_s(0^-) \partial_u G_s(0^-) \} ds \right| > \zeta \Big] = 0, \end{aligned}$$

for every $\zeta > 0$. Since the boundary integrals and the integral involving $\chi(\rho_s)$ are not defined in the whole Skorohod space $\mathcal{D}_{\mathcal{M}}$, we cannot use directly Portman-teau's Theorem to obtain the claim above. To overcome this technical obstacle, fix $\varepsilon > 0$, which will be taken small later. Recall (3.2). Adding and subtracting the convolution of $\rho_t(u)$ with ι_ε , we bound the probability above by the sum of the probabilities

$$\begin{aligned} \mathbb{Q}_*^H \left[\pi. : \sup_{0 \leq t \leq T} \left| \langle \rho_t, G_t \rangle - \langle \rho_0, G_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta) G_s \rangle ds \right. \right. \\ \left. - 2 \int_0^t \langle \chi(\rho_s * \iota_\varepsilon), \partial_u H_s \partial_u G_s \rangle ds - \int_0^t \varphi_s(\rho * \iota_\varepsilon, H) \delta G_s(0) ds \right. \\ \left. - \int_0^t \{ (\rho_s * \iota_\varepsilon)(0^+) \partial_u G_s(0^+) - (\rho_s * \iota_\varepsilon)(0^-) \partial_u G_s(0^-) \} ds \right| > \zeta/4 \Big], \end{aligned} \quad (4.8)$$

$$\mathbb{Q}_*^H \left[\pi. : \sup_{0 \leq t \leq T} \left| 2 \int_0^t \langle \chi(\rho_s * \iota_\varepsilon) - \chi(\rho_s), \partial_u H_s \partial_u G_s \rangle ds \right| > \zeta/4 \right], \quad (4.9)$$

$$\begin{aligned} \mathbb{Q}_*^H \left[\pi. : \sup_{0 \leq t \leq T} \left| \int_0^t \left\{ [(\rho_s * \iota_\varepsilon)(0^+) - \rho_s(0^+)] \partial_u G_s(0^+) \right. \right. \right. \\ \left. \left. - [(\rho_s * \iota_\varepsilon)(0^-) - \rho_s(0^-)] \partial_u G_s(0^-) \right\} ds \right| > \zeta/4 \Big], \end{aligned} \quad (4.10)$$

and

$$\mathbb{Q}_*^H \left[\pi \cdot : \sup_{0 \leq t \leq T} \left| \int_0^t [\varphi_s(\rho * \iota_\varepsilon, H) - \varphi_s(\rho, H)] \delta G_s(0) ds \right| > \zeta/4 \right]. \quad (4.11)$$

By the Proposition 4.4, the sets in (4.9), (4.10) and (4.11) decrease to sets of null probability as $\varepsilon \downarrow 0$. It remains to deal with (4.8). By Portmanteau's Theorem, (4.8) is bounded from above by

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{Q}_{\mu_N}^H \left[\pi \cdot : \sup_{0 \leq t \leq T} \left| \langle \pi_t, G_t \rangle - \langle \pi_0, G_0 \rangle - \int_0^t \langle \pi_s, (\partial_s + \Delta) G_s \rangle ds \right. \right. \\ & \quad - 2 \int_0^t \langle \chi(\pi_s * \iota_\varepsilon), \partial_u H_s \partial_u G_s \rangle ds - \int_0^t \varphi_s(\pi * \iota_\varepsilon, H) \delta G_s(0) ds \\ & \quad \left. \left. - \int_0^t \{ (\pi_s * \iota_\varepsilon)(0^+) \partial_u G_s(0^+) - (\pi_s * \iota_\varepsilon)(0^-) \partial_u G_s(0^-) \} ds \right| > \zeta/4 \right]. \end{aligned}$$

Recalling the identity (3.3), the definition of $\varphi_s(\cdot, H)$ given in (2.12), and the definition of $\mathbb{Q}_{\mu_N}^H$, we can rewrite the previous expression as

$$\begin{aligned} & \lim_{N \rightarrow \infty} \mathbb{P}_{\mu_N}^H \left[\sup_{0 \leq t \leq T} \left| \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, (\partial_s + \Delta) G_s \rangle ds \right. \right. \\ & \quad - 2 \int_0^t \langle \chi(\pi_s^N * \iota_\varepsilon), \partial_u H_s \partial_u G_s \rangle ds - \int_0^t \left\{ \eta_s^{\varepsilon N}(0) \partial_u G_s(0^+) - \eta_s^{\varepsilon N}(-1) \partial_u G_s(0^-) \right\} ds \\ & \quad \left. \left. - \int_0^t \left\{ \eta_s^{\varepsilon N}(-1)(1 - \eta_s^{\varepsilon N}(0))e^{\delta H_s(0)} - \eta_s^{\varepsilon N}(0)(1 - \eta_s^{\varepsilon N}(-1))e^{-\delta H_s(0)} \right\} \delta G_s(0) ds \right| > \frac{\zeta}{4} \right]. \end{aligned}$$

Adding and subtracting $N^2 L_{N,s}^H \langle \pi_s^N, G_s \rangle$, we bound the previous probability by the sum of

$$\overline{\lim}_{N \rightarrow \infty} \mathbb{P}_{\mu_N}^H \left[\sup_{0 \leq t \leq T} \left| \langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle - \int_0^t \langle \pi_s^N, \partial_s G_s \rangle + N^2 L_{N,s}^H \langle \pi_s^N, G_s \rangle ds \right| > \frac{\zeta}{8} \right] \quad (4.12)$$

and

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \mathbb{P}_{\mu_N}^H \left[\sup_{0 \leq t \leq T} \left| \int_0^t N^2 L_{N,s}^H \langle \pi_s^N, G_s \rangle ds - \int_0^t \langle \pi_s^N, \Delta G_s \rangle ds \right. \right. \\ & \quad - 2 \int_0^t \langle \chi(\pi_s^N * \iota_\varepsilon), \partial_u H_s \partial_u G_s \rangle ds \\ & \quad - \int_0^t \left\{ \eta_s^{\varepsilon N}(0) \partial_u G_s(0^+) - \eta_s^{\varepsilon N}(-1) \partial_u G_s(0^-) \right\} ds \\ & \quad - \int_0^t \left\{ \eta_s^{\varepsilon N}(-1)(1 - \eta_s^{\varepsilon N}(0))e^{\delta H_s(0)} \right. \\ & \quad \left. \left. - \eta_s^{\varepsilon N}(0)(1 - \eta_s^{\varepsilon N}(-1))e^{-\delta H_s(0)} \right\} \delta G_s(0) ds \right| > \zeta/8 \right]. \end{aligned} \quad (4.13)$$

The expression inside the first probability is the martingale $M_{N,t}^H(G)$ defined in (4.1). Since its quadratic variations goes to zero, by Doob's inequality, the limit (4.12) is zero. By the formula (4.2) for $N^2 L_{N,s}^H \langle \pi_s^N, G_s \rangle$ and a few applications of Proposition 4.3 we obtain that (4.13) is null, proving the claim (4.7).

Proposition 4.5. *Fix a measurable profile $\rho_0 : \mathbb{T} \rightarrow [0, 1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of probability measures on $\{0, 1\}^{\mathbb{T}^N}$ associated to ρ_0 in the sense of (2.3). Then any limit point of $\mathbb{Q}_{\mu_N}^H$ will be concentrated on absolutely continuous paths $\pi_t(du) = \rho(t, u)du$, with positive density ρ_t bounded by 1, such that ρ is a weak solution of (2.11) with initial condition ρ_0 .*

Proof. Let $\{G_i : i \geq 1\}$ be a countable dense set of functions on $C^{1,2}([0, T] \times [0, 1])$, with respect to the norm $\|G\|_\infty + \|\partial_u G\|_\infty + \|\partial_u^2 G\|_\infty$. Provided by (4.7) and intercepting a countable number of sets of probability one, we can extend (4.7) for all functions $G \in C^{1,2}([0, T] \times [0, 1])$ simultaneously. \square

5. Large deviations upper bound

The proof of the large deviations upper bound is constructed by an optimization over a class of mean-one positive martingales, which must be functions of the process, or, as in our case, close to functions of the process. In the Section 4.2 we have obtained a good candidate to be the mean-one positive martingale, the Radon-Nikodym derivative of the measure $\mathbb{P}_{\mu_N}^H$ with respect to \mathbb{P}_{μ_N} . Since $d\mathbb{P}_{\mu_N}^H/d\mathbb{P}_{\mu_N}$ is not a function of the empirical measure, the first step is to show that it is superexponentially close to a function of the empirical measure.

5.1. Radon-Nikodym derivative (continuation)

To write (4.5) in a simpler form, let us introduce some notation. Given $H \in C^{1,2}([0, T] \times [0, 1])$, consider the linear functional $\ell_H^{int} : \mathcal{D}_{\mathcal{M}} \rightarrow \mathbb{R}$

$$\ell_H^{int}(\pi) = \langle \pi_T, H_T \rangle - \langle \pi_0, H_0 \rangle - \int_0^T \langle \pi_t, (\partial_t + \Delta)H_t \rangle dt. \quad (5.1)$$

With this notation and recalling (3.4) and (3.5), we can rewrite $d\mathbb{P}_{\mu_N}^H/d\mathbb{P}_{\mu_N}$ as

$$\begin{aligned} & \exp \left\{ N \left[\ell_H^{int}(\pi^N) - \int_0^T \frac{1}{2N} \sum_{x \neq -1} \{ \tau_x g_1(\eta_t) + \tau_x g_2(\eta_t) \} (\partial_u H_t)^2 \left(\frac{x}{N} \right) dt \right. \right. \\ & \quad - \int_0^T \left\{ \eta_t(0) \partial_u H_t \left(\frac{0}{N} \right) - \eta_t(-1) \partial_u H_t \left(\frac{-1}{N} \right) \right\} dt \\ & \quad \left. \left. - \int_0^T \left\{ \tau_{-1} g_1(\eta_t) (e^{\delta_N H_{-1}} - 1) + \tau_0 g_2(\eta_t) (e^{-\delta_N H_{-1}} - 1) \right\} dt \right] + N O_{H,T} \left(\frac{1}{N} \right) \right\}. \end{aligned} \quad (5.2)$$

We begin by defining a set where the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ is close to a function of the empirical measure. Consider

$$\begin{aligned} W_{N,\varepsilon}^1(t, \eta) &:= V_{N,\varepsilon}^{F_1, F_2}(t, \eta), & W_{N,\varepsilon}^2(t, \eta) &:= V_{N,\varepsilon}^{G_1, G_2}(t, \eta), \\ W_{N,\varepsilon}^3(t, \eta) &:= \hat{V}_{N,\varepsilon}^{\partial_u H, -1}(t, \eta), & W_{N,\varepsilon}^4(t, \eta) &:= \hat{V}_{N,\varepsilon}^{\partial_u H, 0}(t, \eta), \end{aligned}$$

where V and \hat{V} have been defined in Proposition 3.4 and Corollary 3.6 considering $F_1(t, u) = \frac{1}{2}(\partial_u H_t)^2(u)$, $F_2(t, \frac{-1}{N}) = e^{\delta_N H^{-1}} - 1$, $G_1(t, u) = \frac{1}{2}(\partial_u H_t)^2(u)$ and $G_2(t, \frac{-1}{N}) = e^{-\delta_N H^{-1}} - 1$. Define the set

$$B_{\delta, \varepsilon}^H = \left\{ \eta \in \mathcal{D}_{\Omega_N} ; \left| \int_0^T W_{N,\varepsilon}^i(t, \eta_t) dt \right| \leq \delta, i = 1, 2, 3, 4 \right\}. \quad (5.3)$$

From Proposition 3.4 and Corollary 3.6, this set $B_{\delta, \varepsilon}^H$ has probability superexponentially close to one, i.e., for each $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[(B_{\delta, \varepsilon}^H)^c \right] = -\infty. \quad (5.4)$$

In view of identity (3.3) and expression (5.2), restricted to the set $B_{\delta, \varepsilon}^H$ the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ is equal to

$$\begin{aligned} & \exp \left\{ N \left[\ell_H^{int}(\mathcal{A}) + O_{H,T}(\frac{1}{N}) + O(\delta) \right. \right. \\ & - \int_0^T \frac{1}{2N} \sum_{x \neq -1} \left\{ \tilde{g}_1 \left(\mathcal{A}(\frac{x}{N}), \mathcal{A}(\frac{x+1}{N}) \right) + \tilde{g}_2 \left(\mathcal{A}(\frac{x}{N}), \mathcal{A}(\frac{x+1}{N}) \right) \right\} (\partial_u H_t)^2(\frac{x}{N}) dt \\ & - \int_0^T \left[\mathcal{A}(\frac{0}{N}) \partial_u H_t(\frac{0}{N}) - \mathcal{A}(\frac{-1}{N}) \partial_u H_t(\frac{-1}{N}) \right] dt \\ & - \int_0^T \tilde{g}_1 \left(\mathcal{A}(\frac{-1}{N}), \mathcal{A}(\frac{0}{N}) \right) (e^{\delta_N H^{-1}} - 1) dt \\ & \left. \left. - \int_0^T \tilde{g}_2 \left(\mathcal{A}(\frac{-1}{N}), \mathcal{A}(\frac{0}{N}) \right) (e^{-\delta_N H^{-1}} - 1) dt \right] \right\}, \end{aligned} \quad (5.5)$$

where $\mathcal{A} = \pi_t^N * \iota_\varepsilon$. At this point we have a function of the empirical measure modulo some small errors. Unfortunately, this is not enough to handle with limits on boundary terms. The reason is simple, the convolution $\pi^N * \iota_\varepsilon$ is a function (not a measure anymore) but not a *smooth* function, therefore not necessarily possessing well-behaved side limits. Hence, the next step is to replace $\pi^N * \iota_\varepsilon$ by $(\pi^N * \iota_\gamma^s) * \iota_\varepsilon$, where ι_γ^s is a smooth approximation of identity to be defined next. Notice that ι_γ^s shall not be misunderstood with ι_ε defined in (3.2).

Fix $f : \mathbb{T} \rightarrow \mathbb{R}_+$ a continuous function with support contained in $[-\frac{1}{4}, \frac{1}{4}]$, $0 \leq f \leq 4$, $f(0) > 0$, $\int f = 1$ and symmetric around zero, in other words, satisfying $f(u) = f(1-u)$ for all $u \in \mathbb{T}$. Define the continuous approximation of identity ι_γ^s by $\iota_\gamma^s(u) = \frac{1}{\gamma} f(\frac{u}{\gamma})$.

At this point, we need some approximation estimates to be presented in three next lemmas. Recall that ℓ_H^{int} is the linear functional defined in (5.1).

Lemma 5.1. $|(\pi_t^N * \iota_\varepsilon)(v) - ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(v)| \leq \frac{\gamma}{\varepsilon}$, uniformly in $v \in \mathbb{T}$, $N \in \mathbb{N}$, and $t \in [0, T]$.

Proof. Writing the expression $|(\pi_t^N * \iota_\varepsilon)(v) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon)(v)|$ as

$$\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \iota_\varepsilon\left(\frac{x}{N}, v\right) - \int_{\mathbb{T}} \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \iota_\gamma(u - \frac{x}{N}) \iota_\varepsilon(u, v) du \right|.$$

Using the rule of maximum of one particle per site, the last expression is bounded by

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N} \left| \iota_\varepsilon\left(\frac{x}{N}, v\right) - \int_{\mathbb{T}} \iota_\gamma(u - \frac{x}{N}) \iota_\varepsilon(u, v) du \right|.$$

Fix N , v and ε , then $\iota_\varepsilon(\cdot, v)$ is the indicator function of an open interval $(z, z + \varepsilon)$, for $z = v$ or $z = 1 - \varepsilon$. The summand above is possibly not zero only if $\frac{x}{N}$ belongs to the open intervals $(z - \frac{\gamma}{4}, z + \frac{\gamma}{4})$ or $(z + \varepsilon - \frac{\gamma}{4}, z + \varepsilon + \frac{\gamma}{4})$. The summands are bounded by $\frac{1}{\varepsilon}$, and the number of non zero summands is of order γN , which concludes the proof. \square

Lemma 5.2. $\ell_H^{int}(\pi^N) = \ell_H^{int}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon) + O_H(\varepsilon) + O_H(\frac{\gamma}{\varepsilon})$, uniformly in $N \in \mathbb{N}$.

Proof. First we compare $\ell_H^{int}((\pi^N * \iota_\gamma) * \iota_\varepsilon)$ with $\ell_H^{int}(\pi^N * \iota_\varepsilon)$. Using the Lemma 5.1, we obtain the difference between this functions is

$$\begin{aligned} & \left| \left\langle ((\pi_T^N * \iota_\gamma) * \iota_\varepsilon) - (\pi_T^N * \iota_\varepsilon), H_T \right\rangle - \left\langle ((\pi_0^N * \iota_\gamma) * \iota_\varepsilon) - (\pi_0^N * \iota_\varepsilon), H_0 \right\rangle \right. \\ & \quad \left. - \int_0^T \left\langle ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon) - (\pi_t^N * \iota_\varepsilon), (\partial_t + \Delta) H_t \right\rangle dt \right| \leq C(H) \frac{\gamma}{\varepsilon}. \end{aligned}$$

Then, we need only analyze the expression below

$$\begin{aligned} & \left| \ell_H^{int}(\pi^N * \iota_\varepsilon) - \ell_H^{int}(\pi^N) \right| = \left| \left\langle (\pi_T^N * \iota_\varepsilon) - \pi_T^N, H_T \right\rangle - \left\langle (\pi_0^N * \iota_\varepsilon) - \pi_0^N, H_0 \right\rangle \right. \\ & \quad \left. - \int_0^T \left\langle (\pi_t^N * \iota_\varepsilon) - \pi_t^N, (\partial_t + \Delta) H_t \right\rangle dt \right|. \end{aligned}$$

We handle only the first term, because the others terms are similar. Thus,

$$\begin{aligned} \left\langle (\pi_t^N * \iota_\varepsilon), H_t \right\rangle &= \int_{\mathbb{T}} (\pi_t^N * \iota_\varepsilon)(v) H_t(v) dv = \int_{\mathbb{T}} \frac{1}{N} \sum_{y \in \mathbb{T}_N} \eta_t(y) \iota_\varepsilon^a\left(\frac{y}{N}, v\right) H_t(v) dv \\ &= \frac{1}{N} \sum_{y \in \mathbb{T}_N} \eta_t(y) \int_{\mathbb{T}} H_t(v) \iota_\varepsilon^a\left(\frac{y}{N}, v\right) dv = \langle \pi_t^N, H_t \rangle + O_H(\varepsilon). \end{aligned}$$

This approximation holds uniformly in time and N , since $H \in C^{1,2}([0, T] \times (0, 1))$ and there is at most one particle per site. Therefore,

$$|\ell_H^{int}(\pi^N * \iota_\varepsilon) - \ell_H^{int}(\pi^N)| = O_H(\varepsilon).$$

□

Lemma 5.3. *The function $\left| \tilde{g}_i \left((\pi_t^N * \iota_\varepsilon) \left(\frac{x}{N} \right), (\pi_t^N * \iota_\varepsilon) \left(\frac{x+1}{N} \right) \right) - \tilde{g}_i \left(((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon) \left(\frac{x}{N} \right), ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon) \left(\frac{x+1}{N} \right) \right) \right|$ is $O(\frac{2}{\varepsilon})$ for $i = 1, 2$.*

Proof. This proof follows by the definition of \tilde{g}_1 and \tilde{g}_2 (see (3.4) and (3.5)), the triangular inequality and the Lemma 5.1. □

Lemmas 5.1, 5.2 and 5.3 allow to replace π_t^N by $(\pi_t^N * \iota_\gamma^s)$ in the expression of Radon-Nikodym derivative (5.5) modulus small errors. Hence, restricted to the set $B_{\delta, \varepsilon}^H$, the Radon-Nikodym derivative $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ becomes

$$\begin{aligned} & \exp \left\{ N \left[\ell_H^{int}(\mathcal{B}) + O_{H,T}(\frac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\frac{\gamma}{\varepsilon}) \right. \right. \\ & - \int_0^T \frac{1}{2N} \sum_{x \neq -1} \left\{ \tilde{g}_1 \left(\mathcal{B} \left(\frac{x}{N} \right), \mathcal{B} \left(\frac{x+1}{N} \right) \right) + \tilde{g}_2 \left(\mathcal{B} \left(\frac{x}{N} \right), \mathcal{B} \left(\frac{x+1}{N} \right) \right) \right\} (\partial_u H_t)^2 \left(\frac{x}{N} \right) dt \\ & - \int_0^T \left[\mathcal{B} \left(\frac{0}{N} \right) \partial_u H_t \left(\frac{0}{N} \right) - \mathcal{B} \left(\frac{-1}{N} \right) \partial_u H_t \left(\frac{-1}{N} \right) \right] dt \\ & - \int_0^T \tilde{g}_1 \left(\mathcal{B} \left(\frac{-1}{N} \right), \mathcal{B} \left(\frac{0}{N} \right) \right) (e^{\delta_N H_{-1}} - 1) dt \\ & \left. - \int_0^T \tilde{g}_2 \left(\mathcal{B} \left(\frac{-1}{N} \right), \mathcal{B} \left(\frac{0}{N} \right) \right) (e^{-\delta_N H_{-1}} - 1) dt \right] \right\}, \end{aligned} \quad (5.6)$$

where $\mathcal{B} = (\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon$. Recall $\chi(u) = u(1-u)$. The next three lemmas allow to replace the sum involving \tilde{g}_i by an integral in χ and to make a little adjustment at the boundaries.

Lemma 5.4. *The difference*

$$\begin{aligned} & \left| \frac{1}{N} \sum_{x \neq -1} \tilde{g}_i \left(((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon) \left(\frac{x}{N} \right), ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon) \left(\frac{x+1}{N} \right) \right) (\partial_u H_t)^2 \left(\frac{x}{N} \right) \right. \\ & \left. - \int_{\mathbb{T}} \chi \left(((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(v) \right) (\partial_u H_t)^2(v) dv \right|, \end{aligned}$$

can be denoted by some function $R_N^1(H, t, \varepsilon, \gamma)$, which goes to zero, when $N \rightarrow \infty$, uniformly in $t \in [0, T]$, with $i = 1, 2$.

Proof. Consider $i = 1$. To simplify notation, denote

$$f^N \left(\frac{x}{N} \right) := \tilde{g}_1 \left(((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon) \left(\frac{x}{N} \right), ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon) \left(\frac{x+1}{N} \right) \right)$$

and

$$g^N(v) := \tilde{g}_1\left((\pi_t^N * \iota_\gamma) * \iota_\varepsilon(v), (\pi_t^N * \iota_\gamma) * \iota_\varepsilon(v)\right) = \chi\left((\pi_t^N * \iota_\gamma) * \iota_\varepsilon(v)\right).$$

From the definition of ι_ε , if $x \neq a_N$,

$$|(\varrho * \iota_\varepsilon)(\frac{x}{N}) - (\varrho * \iota_\varepsilon)(v)| \leq \frac{\|\varrho\|_\infty}{\varepsilon N}, \quad \forall v \in [\frac{x}{N}, \frac{x+1}{N}],$$

where ϱ is any bounded function defined on the torus. The same inequality is still valid with $x+1$ replacing x in left side of inequality. Since $\|\pi_t^N * \iota_\gamma\|_\infty \leq 4$, if $x \neq a_N$,

$$|f^N(\frac{x}{N}) - g^N(v)| = O(\frac{1}{\varepsilon N}), \quad \forall v \in [\frac{x}{N}, \frac{x+1}{N}].$$

Then,

$$\begin{aligned} & \left| \frac{1}{N} \sum_{x \neq a_N} f^N(\frac{x}{N}) (\partial_u H_t)^2(\frac{x}{N}) - \int_{\mathbb{T}} g^N(v) (\partial_u H_t)^2(v) dv \right| \\ & \leq \left| \frac{1}{N} \sum_{x \neq a_N} f^N(\frac{x}{N}) \left[(\partial_u H_t)^2(\frac{x}{N}) - N \int_{\mathbb{T}} \mathbf{1}_{[\frac{x}{N}, \frac{x+1}{N})}(v) (\partial_u H_t)^2(v) dv \right] \right| \\ & + \left| \sum_{x \neq a_N} \int_{\mathbb{T}} \mathbf{1}_{[\frac{x}{N}, \frac{x+1}{N})}(v) \left[f^N(\frac{x}{N}) - g^N(v) \right] (\partial_u H_t)^2(v) dv \right| \\ & + \left| \int_{\mathbb{T}} \mathbf{1}_{[\frac{a_N}{N}, \frac{a_N+1}{N})}(v) g^N(v) (\partial_u H_t)^2(v) dv \right| \\ & \leq \frac{1}{N} \sum_{x \neq a_N} \left| (\partial_u H_t)^2(\frac{x}{N}) - N \int_{\mathbb{T}} \mathbf{1}_{[\frac{x}{N}, \frac{x+1}{N})}(v) (\partial_u H_t)^2(v) dv \right| \\ & + O(\frac{1}{\varepsilon N}) \int_{\mathbb{T}} |(\partial_u H_t)^2(v)| dv + \frac{1}{N} \|(\partial_u H_t)^2\|_\infty. \end{aligned}$$

Since H belongs to $C^{1,2}([0, T] \times \overline{(0, 1)})$, the first sum goes to zero, when $N \rightarrow \infty$, which finishes the proof. \square

Lemma 5.5. Denote by $R_N^2(H, t, \varepsilon, \gamma)$ the following expression:

$$\begin{aligned} & \left| ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(\frac{0}{N}) \partial_u H_t(\frac{0}{N}) - ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(\frac{-1}{N}) \partial_u H_t(\frac{-1}{N}) \right. \\ & \left. - ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(0^+) \partial_u H_t(0^+) - ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(0^-) \partial_u H_t(0^-) \right|. \end{aligned}$$

Then $R_N^2(H, t, \varepsilon, \gamma)$ goes to zero, when N increases to ∞ , uniformly in $t \in [0, T]$.

Proof. This proof follows by fact that $\iota_\varepsilon(\cdot, \frac{-1}{N})$, $\iota_\varepsilon(\cdot, \frac{0}{N})$, $\partial_u H_t(\frac{-1}{N})$ and $\partial_u H_t(\frac{0}{N})$ converges to $\iota_\varepsilon(\cdot, 0^-)$, $\iota_\varepsilon(\cdot, 0^+)$, $\partial_u H_t(0^-)$ and $\partial_u H_t(0^+)$, respectively, as N increases to infinity. \square

Lemma 5.6. *The expression below*

$$\begin{aligned} & \left| \tilde{g}_1 \left(((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(0^-), ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(0^+) \right) (e^{\delta H_t(0)} - 1) \right. \\ & \quad \left. - \tilde{g}_1 \left(((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(\frac{-1}{N}), ((\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon)(\frac{0}{N}) \right) (e^{\delta_N H^{-1}} - 1) \right| \end{aligned}$$

is a function $R_N^3(H, t, \varepsilon, \gamma)$, which goes to zero, when N increases to ∞ , uniformly in $t \in [0, T]$. Analogous statement for \tilde{g}_2 .

Proof. We only analyze the first statement, the second one is just the same argument. By definition of \tilde{g}_1 , the expression in the left side of the first equality is bounded above by

$$\begin{aligned} & \left| ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon)(\frac{-1}{N}) (e^{\nabla_N H^{-1}} - 1) - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon)(0^-) (e^{H_t(0^+) - H_t(0^-)} - 1) \right| \\ & + \left| ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon)(\frac{-1}{N}) ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon)(\frac{0}{N}) (e^{\nabla_N H^{-1}} - 1) \right. \\ & \quad \left. - ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon)(0^-) ((\pi_t^N * \iota_\gamma) * \iota_\varepsilon)(0^+) (e^{H_t(0^+) - H_t(0^-)} - 1) \right|. \end{aligned}$$

The conclusion follows by fact that $\iota_\varepsilon(\cdot, \frac{-1}{N})$, $\iota_\varepsilon(\cdot, \frac{0}{N})$ and $e^{\nabla_N H^{-1}} - 1$ converges to $\iota_\varepsilon(\cdot, 0^-)$, $\iota_\varepsilon(\cdot, 0^+)$ and $e^{H_t(0^+) - H_t(0^-)} - 1$, respectively, as N increases to infinity. \square

Denote $R_N(H, T, \varepsilon, \gamma)$ the errors from the lemmas 5.4, 5.5 and 5.6, notice that

$$\lim_{N \rightarrow \infty} R_N(H, T, \varepsilon, \gamma) = 0. \quad (5.7)$$

By means of these lemmas, we can rewrite the expression (5.6) of the Radon-Nikodym derivative $\mathbf{dP}_{\mu_N}^H / \mathbf{dP}_{\mu_N}$ on the set $B_{\delta, \varepsilon}^H$ as

$$\begin{aligned} & \exp \left\{ N \left[\ell_H^{int}(\mathcal{B}) - \int_0^T \int_{\mathbb{T}} \chi(\mathcal{B}(v)) (\partial_u H_t)^2(v) dv dt \right. \right. \\ & \quad - \int_0^T \left[\mathcal{B}(0^+) \partial_u H_t(0^+) - \mathcal{B}(0^-) \partial_u H_t(0^-) \right] dt \\ & \quad - \int_0^T \tilde{g}_1(\mathcal{B}(0^-), \mathcal{B}(0^+)) (e^{\delta H_t(0)} - 1) dt \\ & \quad - \int_0^T \tilde{g}_2(\mathcal{B}(0^-), \mathcal{B}(0^+)) (e^{-\delta H_t(0)} - 1) dt \\ & \quad \left. \left. + R_N(H, T, \varepsilon, \gamma) + O(\delta) + O_H(\varepsilon) + O_H(\frac{\gamma}{\varepsilon}) \right] \right\}, \end{aligned} \quad (5.8)$$

where $\mathcal{B} = (\pi_t^N * \iota_\gamma^s) * \iota_\varepsilon$ as before.

Now, we observe that the functional ℓ_H defined in (2.16) and the functional ℓ_H^{int} given in Definition (5.1) are related by

$$\begin{aligned} \ell_H(\pi) &= \ell_H^{int}(\pi) - \int_0^T \{\rho_t(0^+) \partial_u H_t(0^+) - \rho_t(0^-) \partial_u H_t(0^-)\} dt \\ &\quad + \int_0^T (\rho_t(0^+) - \rho_t(0^-))(H_t(0^+) - H_t(0^-)) dt. \end{aligned}$$

Moreover, because of its smoothness, $(\pi^N * \iota_\gamma^s) * \iota_\varepsilon$ has finite energy, see Definition 2.11. Recalling Definition 2.12 of the functional J_H , and expression (5.8), we conclude that $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ restricted to $B_{\delta, \varepsilon}^H$ is

$$\exp \left\{ N \left[J_H \left((\pi^N * \iota_\gamma^s) * \iota_\varepsilon \right) + R_N(H, T, \varepsilon, \gamma) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right) \right] \right\}. \quad (5.9)$$

Let us proceed to the next step. It is not difficult to see that the set $\{\pi \in \mathcal{D}_{\mathcal{M}} ; \mathcal{E}(\pi) < \infty\}$ is not closed in the concerning topology (the Skorohod topology on $\mathcal{D}_{\mathcal{M}}$). This is an obstacle in order to apply the Minimax Lemma, see [KL, Lemma 3.3, page 364], which is an important device in the proof of the large deviations upper bound. To invoke the Minimax Lemma, the functional J_H should be lower semi-continuous⁴, what is not true precisely because the set $\{\pi \in \mathcal{D}_{\mathcal{M}} ; \mathcal{E}(\pi) < \infty\}$ is not closed.

To overcome this obstacle, we begin by introducing the next sets.

Definition 5.7. Let $A_{k,l}$, $A_{k,l}^\varepsilon$, and $A_{k,l}^{\varepsilon, \gamma}$ be the subsets of trajectories given by

$$\begin{aligned} A_{k,l} &= \{\pi \in \mathcal{D}_{\mathcal{M}} ; \max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi) \leq l\}, \\ A_{k,l}^\varepsilon &= \{\pi \in \mathcal{D}_{\mathcal{M}} ; \pi * \iota_\varepsilon \in A_{k,l}\}, \\ A_{k,l}^{\varepsilon, \gamma} &= \{\pi \in \mathcal{D}_{\mathcal{M}} ; (\pi * \iota_\gamma^s) * \iota_\varepsilon \in A_{k,l}\}. \end{aligned}$$

Proposition 5.8. For fixed $\varepsilon, \gamma, k, l$, the set $A_{k,l}^{\varepsilon, \gamma}$ is closed.

Proof. It is sufficient to show that the function $\psi : \mathcal{D}_{\mathcal{M}} \rightarrow \overline{\mathbb{R}}$ given by $\psi(\pi) = \mathcal{E}_{H_j}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon)$ is continuous. Let $\{\pi_t^n ; t \in [0, T]\}_n$ converging to $\{\pi_t ; t \in [0, T]\}$ on $\mathcal{D}_{\mathcal{M}}$. Therefore, $\pi_t^n \xrightarrow{\omega^*} \pi_t$, almost surely in time. For such t , $\pi_t * \iota_\gamma^s = \lim_{n \rightarrow \infty} \pi_t^n * \iota_\gamma^s$, since ι_γ^s is a continuous function. By the Dominated Convergence Theorem,

$$((\pi_t * \iota_\gamma^s) * \iota_\varepsilon)(v) = \int_{\mathbb{T}} \lim_{n \rightarrow \infty} (\pi_t^n * \iota_\gamma^s)(u) \iota_\varepsilon(u, v) du = \lim_{n \rightarrow \infty} ((\pi_t^n * \iota_\gamma^s) * \iota_\varepsilon)(v). \quad (5.10)$$

⁴About signs and conventions: in [KL, Lemma 3.3, page 364] the statement is about an upper continuous functional, but the functional J_β appearing there corresponds to minus our functional J_H here.

Again by the Dominated Convergence Theorem,

$$\begin{aligned} \left\langle \partial_u H_j, (\pi_t * \iota_\gamma^s) * \iota_\varepsilon \right\rangle &= \int_0^T \int_{\mathbb{T}} \partial_u H_j(t, v) ((\pi_t * \iota_\gamma^s) * \iota_\varepsilon)(v) dv dt \\ &= \lim_{n \rightarrow \infty} \int_0^T \int_{\mathbb{T}} \partial_u H_j(t, v) ((\pi_t^n * \iota_\gamma^s) * \iota_\varepsilon)(v) dv dt \\ &= \lim_{n \rightarrow \infty} \left\langle \partial_u H_j, (\pi^n * \iota_\gamma^s) * \iota_\varepsilon \right\rangle. \end{aligned}$$

□

Proposition 5.9. *For fixed k , and l ,*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{\gamma \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\pi^N \in (A_{k,l}^{\varepsilon, \gamma})^c \right] \leq -l + K_0 T.$$

Proof. For all $r > 0$,

$$\begin{aligned} \mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon) \geq l \right] &\leq \mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi^N * \iota_\varepsilon) \geq l - r \right] \\ &\quad + \mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon - \pi^N * \iota_\varepsilon) \geq r \right]. \end{aligned}$$

By Lemma 5.1, we have that

$$\max_{1 \leq j \leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon - \pi^N * \iota_\varepsilon) \leq \max_{1 \leq j \leq k} \left\langle \partial_u H_j, (\pi^N * \iota_\gamma^s) * \iota_\varepsilon - \pi^N * \iota_\varepsilon \right\rangle \leq \frac{C\gamma}{\varepsilon},$$

where $C = C(\{H\}_{1 \leq j \leq k})$. Therefore,

$$\mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon) \geq l \right] \leq \mathbb{P}_{\mu_N} \left[\frac{C\gamma}{\varepsilon} \geq r \right],$$

which is zero for γ small enough. Hence,

$$\begin{aligned} \overline{\lim}_{\gamma \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon) \geq l \right] \\ \leq \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi^N * \iota_\varepsilon) \geq l - r \right]. \end{aligned}$$

By Corollary 3.9, we get

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{\gamma \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N} \left[\max_{1 \leq j \leq k} \mathcal{E}_{H_j}((\pi^N * \iota_\gamma^s) * \iota_\varepsilon) \geq l \right] \leq -l + K_0 T + r.$$

Since r is arbitrary, the proof is finished. □

In (5.9) appears the term $(\pi^N * \iota_\gamma^s) * \iota_\varepsilon$ and we would like to take $\gamma \downarrow 0$ and $\varepsilon \downarrow 0$. To avoid technical problems that would come into scene from the fact π_t^N does not have density with respect to the Lebesgue measure, we define below another family of sets.

Fix a sequence $\{F_i\}_{i \geq 1}$ of smooth non negative functions dense in the subset of non-negative functions $C(\mathbb{T})$ with respect to the uniform topology. For $i \geq 1$ and $j \geq 1$, we define the set

$$D_i^j = \left\{ \pi \in \mathcal{D}_{\mathcal{M}} ; 0 \leq \langle \pi_t, F_i \rangle \leq \int_{\mathbb{T}} F_i(u) du + \frac{1}{j} \|F_i'\|_{\infty}, 0 \leq t \leq T \right\}, \quad (5.11)$$

and for $m \geq 1$ and $j \geq 1$, let $E_m^j = \bigcap_{i=1}^m D_i^j$.

Proposition 5.10. *It holds:*

- (i) Given $i \geq 1$ and $j \geq 1$, the set D_i^j is a closed subset of $\mathcal{D}_{\mathcal{M}}$;
- (ii) $\mathcal{D}_{\mathcal{M}_0} = \bigcap_{j \geq 1} \bigcap_{m \geq 1} E_m^j$;
- (iii) Given $m \geq 1$ and $j \geq 1$, $\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}[\pi^N \in (E_m^j)^c] = -\infty$.

Proof. (i) Since F_i continuous, the function $\pi \mapsto \sup_{0 \leq t \leq T} \langle \pi_t, F_i \rangle$ is continuous.

(ii) The inclusion $\mathcal{D}_{\mathcal{M}_0} \subset \bigcap_{j \geq 1} \bigcap_{m \geq 1} E_m^j$ is trivial. The inclusion on the other hand follows by approximating indicators functions of open intervals by a suitable sequence in $\{F_i\}_{i \geq 1}$ and in j .

(iii) The probability $\mathbb{P}_{\mu_N}[\pi^N \in (E_m^j)^c]$ is

$$\mathbb{P}_{\mu_N} \left[\bigcup_{i=1}^m \left\{ \frac{1}{N} \sum_{x \in \mathbb{T}_N} F_i\left(\frac{x}{N}\right) \eta_t(x) > \int_{\mathbb{T}} F_i(u) du + \frac{1}{j} \|F_i'\|_{\infty}, \text{ for some } t \in [0, T] \right\} \right].$$

From the elementary inequality

$$\left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} F_i\left(\frac{x}{N}\right) - \int_{\mathbb{T}} F_i(u) du \right| \leq \sum_{x \in \mathbb{T}_N} \int_{[\frac{x}{N}, \frac{x+1}{N})} |F_i\left(\frac{x}{N}\right) - F_i(u)| du \leq \frac{\|F_i'\|_{\infty}}{N},$$

and by the fact that there is at most one particle per site, we conclude that $\mathbb{P}_{\mu_N}[\pi^N \in (E_m^j)^c]$ vanishes for N sufficiently large, concluding the proof. \square

Keeping in mind that $\mathcal{E}((\pi * \iota_{\gamma}^s) * \iota_{\varepsilon}) < \infty$, for all $\pi \in \mathcal{D}_{\mathcal{M}}$, define

$$J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) = \begin{cases} \hat{J}_H\left((\pi * \iota_{\gamma}^s) * \iota_{\varepsilon}\right), & \text{if } \pi \in A_{k, l}^{\zeta, \gamma} \cap E_m^j, \\ +\infty, & \text{otherwise.} \end{cases} \quad (5.12)$$

Finally, $\mathbf{d}\mathbb{P}_{\mu_N}^H / \mathbf{d}\mathbb{P}_{\mu_N}$ restricted to the set $\{\pi^N \in A_{k, l}^{\zeta, \gamma} \cap E_m^j\} \cap B_{\delta, \varepsilon}^H$ is

$$\exp \left\{ N \left[J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi^N) + R_N(H, T, \varepsilon, \gamma) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right) \right] \right\}. \quad (5.13)$$

This is the appropriate form for the Radon-Nikodym derivative to be used in the next section.

5.2. Upper bound for compact sets

We start by studying the upper bound for open sets. Let $\mathcal{O} \subseteq \mathcal{D}_{\mathcal{M}}$ be an open set and fix a function $H \in C^{1,2}([0, T] \times [0, 1])$. Then

$$\begin{aligned} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] &= \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}[\pi^N \in \mathcal{O}] \\ &\leq \max \left\{ \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}[\{\pi^N \in \mathcal{O} \cap A_{k,l}^{\zeta,\gamma} \cap E_m^j\} \cap B_{\delta,\varepsilon}^H], R_k^l(\zeta, \gamma), R_m^j, R_H^\delta(\varepsilon) \right\} \end{aligned}$$

where we have denoted

$$\begin{aligned} R_k^l(\zeta, \gamma) &= \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}[\{\pi^N \in (A_{k,l}^{\zeta,\gamma})^c\}], \\ R_m^j &= \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}[\{\pi^N \in (E_m^j)^c\}], \\ R_H^\delta(\varepsilon) &= \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}[(B_{\delta,\varepsilon}^H)^c]. \end{aligned}$$

By Propositions 5.9 and 5.10 and the limit (5.4), the expressions above satisfy

$$\overline{\lim}_{\zeta \downarrow 0} \overline{\lim}_{\gamma \downarrow 0} R_k^l(\zeta, \gamma) \leq -l + K_0 T, \quad R_m^j = -\infty, \quad \text{and} \quad \overline{\lim}_{\varepsilon \downarrow 0} R_H^\delta(\varepsilon) = -\infty.$$

Transforming the measure by the Radon-Nikodym derivative and recalling its expression (5.13),

$$\begin{aligned} \mathbb{P}_{\mu_N}[\{\pi^N \in \mathcal{O} \cap A_{k,l}^{\zeta,\gamma} \cap E_m^j\} \cap B_{\delta,\varepsilon}^H] &= \mathbb{E}_{\mu_N} \left[\left(\frac{d\mathbb{P}_{\mu_N}^H}{d\mathbb{P}_{\mu_N}} \right)^{-1} \mathbf{1}_{\{\pi^N \in \mathcal{O} \cap A_{k,l}^{\zeta,\gamma} \cap E_m^j\} \cap B_{\delta,\varepsilon}^H} \right] \\ &= \mathbb{E}_{\mu_N} \left[\exp \left\{ N \left[-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi^N) + R_N(H, T, \varepsilon, \gamma) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right) \right] \right\} \mathbf{1}_{\mathbf{D}} \right], \end{aligned}$$

being $\mathbf{D} := \{\pi^N \in \mathcal{O} \cap A_{k,l}^{\zeta,\gamma} \cap E_m^j\} \cap B_{\delta,\varepsilon}^H$. Therefore,

$$\begin{aligned} \frac{1}{N} \log \mathbb{P}_{\mu_N}[\{\pi^N \in \mathcal{O} \cap A_{k,l}^{\zeta,\gamma} \cap E_m^j\} \cap B_{\delta,\varepsilon}^H] \\ \leq \sup_{\pi \in \mathcal{O}} \{-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi)\} + R_N(H, T, \varepsilon, \gamma) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right). \end{aligned}$$

By (5.7), for all $\gamma, \varepsilon, \zeta, \delta > 0$, for all $k, l, m, j \in \mathbb{N}$ and $H \in C^{1,2}([0, T] \times [0, 1])$, we have

$$\begin{aligned} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \\ \leq \max \left\{ \sup_{\pi \in \mathcal{O}} \{-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi)\} + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right), R_k^l(\zeta, \gamma), R_m^j, R_H^\delta(\varepsilon) \right\} \\ = \max \left\{ \sup_{\pi \in \mathcal{O}} \{-J_{H,\gamma,\varepsilon,\zeta}^{k,l,m,j}(\pi)\} + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right), R_k^l(\zeta, \gamma), R_H^\delta(\varepsilon) \right\}. \end{aligned}$$

Since we do not have any restrictions on the parameters, we can optimize over $\gamma, \varepsilon, \zeta, \delta, k, l, m, j, H$, which yields

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \\ & \leq \inf_{\substack{\gamma, \varepsilon, \zeta, \delta, \\ k, l, m, j, H}} \max \left\{ \sup_{\pi \in \mathcal{O}} \left\{ -J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right), R_k^l(\zeta, \gamma), R_H^\delta(\varepsilon) \right\} \right\} \\ & = \inf_{\substack{\gamma, \varepsilon, \zeta, \delta, \\ k, l, m, j, H}} \sup_{\pi \in \mathcal{O}} \max \left\{ -J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right), R_k^l(\zeta, \gamma), R_H^\delta(\varepsilon) \right\}. \end{aligned} \quad (5.14)$$

Proposition 5.11. *For fixed $\gamma, \varepsilon, \zeta, \delta, k, l, m, j, H$, the functional*

$$\max \left\{ -J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right), R_k^l(\zeta, \gamma), R_H^\delta(\varepsilon) \right\}$$

is upper semi-continuous in $\mathcal{D}_{\mathcal{M}}$.

Proof. In the maximum above, the only term that depends on π is $J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi)$. By the Propositions 5.8 and 5.10, it is enough to prove the continuity of $\hat{J}((\pi * \iota_\gamma^s) * \iota_\varepsilon)$ in $\mathcal{D}_{\mathcal{M}}$.

Let $\pi^n \rightarrow \pi$ in the topology of $\mathcal{D}_{\mathcal{M}}$. In particular, π_t^n converges weakly* to π_t in \mathcal{M} , for almost all $t \in [0, T]$. According to (5.10) and iterated applications of Dominated Convergence Theorem we can assure the continuity of $\hat{J}((\pi * \iota_\gamma^s) * \iota_\varepsilon)$. \square

Provided by the proposition above, we may apply the Minimax Lemma [KL, Lemma A2.3.3], interchanging supremum with infimum in (5.14), and passing to compact sets. Then, for all $\mathcal{K} \subset \mathcal{D}_{\mathcal{M}}$ compact,

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{K}] \\ & \leq \sup_{\pi \in \mathcal{K}} \inf_{\substack{\gamma, \varepsilon, \zeta, \delta, \\ k, l, m, j, H}} \max \left\{ -J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) + O(\delta) + O_H(\varepsilon) + O_H\left(\frac{\gamma}{\varepsilon}\right), R_k^l(\zeta, \gamma), R_H^\delta(\varepsilon) \right\}. \end{aligned} \quad (5.15)$$

The next result connects $J_H(\pi)$ and $J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi)$.

Proposition 5.12. *For all $\pi \in \mathcal{D}_{\mathcal{M}}$,*

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{l \rightarrow \infty} \overline{\lim}_{k \rightarrow \infty} \overline{\lim}_{\zeta \downarrow 0} \overline{\lim}_{\gamma \downarrow 0} \overline{\lim}_{j \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) \geq J_H(\pi).$$

Proof. Recall (5.12) and fix $\pi \in \mathcal{D}_{\mathcal{M}}$. We claim that

$$\overline{\lim}_{j \rightarrow \infty} \overline{\lim}_{m \rightarrow \infty} J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) = \begin{cases} \hat{J}_H((\pi * \iota_\gamma^s) * \iota_\varepsilon), & \text{if } \pi \in A_{k, l}^{\zeta, \gamma} \cap \mathcal{D}_{\mathcal{M}_0} \\ +\infty, & \text{otherwise} \end{cases}. \quad (5.16)$$

The equality above derives from the fact that if $\pi \notin \mathcal{D}_{\mathcal{M}_0}$, there exist m and j such that $\pi \notin E_m^j$. To check this, apply the definition of an absolute continuity with respect to the Lebesgue measure. This proves (5.16).

Let us step to the limit in γ . We claim that

$$\overline{\lim}_{\gamma \downarrow 0} \begin{cases} \hat{J}_H((\pi * \iota_\gamma^s) * \iota_\varepsilon), & \text{if } \pi \in A_{k,l}^{\zeta,\gamma} \cap \mathcal{D}_{\mathcal{M}_0} \\ +\infty, & \text{otherwise} \end{cases} \geq \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \pi \in A_{k,l+1}^\zeta \cap \mathcal{D}_{\mathcal{M}_0} \\ +\infty, & \text{otherwise} \end{cases} \quad (5.17)$$

If $\pi \notin A_{k,l}^{\zeta,\gamma} \cap \mathcal{D}_{\mathcal{M}_0}$ for all γ , the inequality (5.17) is obvious. From Definition 5.7, if $\pi \in A_{k,l}^{\zeta,\gamma} \cap \mathcal{D}_{\mathcal{M}_0}$, it is immediate that

$$\max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta) \leq l + \max_{1 \leq j \leq k} \left\langle \left\langle \partial_u H_j, \pi * \iota_\zeta - (\pi * \iota_\gamma^s) * \iota_\zeta \right\rangle \right\rangle.$$

For fixed ζ and k , we can find γ small enough in such a way

$$\max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta) \leq l + 1,$$

implying $\pi \in A_{k,l+1}^\zeta \cap \mathcal{D}_{\mathcal{M}_0}$. Besides, for fixed $\varepsilon > 0$, the double convolution $(\pi * \iota_\gamma^s) * \iota_\varepsilon$ converges uniformly to $\pi * \iota_\varepsilon$, leading to

$$\lim_{\gamma \downarrow 0} \hat{J}_H((\pi * \iota_\gamma^s) * \iota_\varepsilon) = \hat{J}_H(\pi * \iota_\varepsilon)$$

and hence proves (5.17). The ensuing step is to take the limit in $\zeta \downarrow 0$. We claim that

$$\overline{\lim}_{\zeta \downarrow 0} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \pi \in A_{k,l+1}^\zeta \cap \mathcal{D}_{\mathcal{M}_0} \\ +\infty, & \text{otherwise} \end{cases} \geq \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \pi \in A_{k,l+2} \cap \mathcal{D}_{\mathcal{M}_0} \\ +\infty, & \text{otherwise} \end{cases}. \quad (5.18)$$

In fact, if $\pi \in A_{k,l+1}^\zeta \cap \mathcal{D}_{\mathcal{M}_0}$, then

$$\begin{aligned} \max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi) &= \max_{1 \leq j \leq k} \mathcal{E}_{H_j}(\pi * \iota_\zeta) + \max_{1 \leq j \leq k} \left\langle \left\langle \partial_u H_j, \pi - \pi * \iota_\zeta \right\rangle \right\rangle \\ &\leq l + 1 + \max_{1 \leq j \leq k} \int_0^T \int_{\mathbb{T}} \partial_u H_j(t, u) (\rho_t(u) - (\pi_t * \iota_\zeta)(u)) du dt. \end{aligned}$$

By the Lebesgue Differentiation Theorem, it is possible to choose small ζ such that the integral term in the right hand side of above is smaller than 1. This proves (5.18). Taking the limit in $k \rightarrow \infty$ in the right hand side of (5.18), we obtain

$$\overline{\lim}_{k \rightarrow \infty} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \pi \in A_{k,l+2} \cap \mathcal{D}_{\mathcal{M}_0} \\ +\infty, & \text{otherwise} \end{cases} = \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \mathcal{E}(\pi) \leq l + 2 \\ +\infty, & \text{otherwise} \end{cases}, \quad (5.19)$$

because $\{\pi; \mathcal{E}(\pi) \leq l + 2\} \subset \mathcal{D}_{\mathcal{M}_0}$. Next, taking the limit in $l \rightarrow \infty$ in the right hand side of (5.19), we get

$$\overline{\lim}_{l \rightarrow \infty} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \mathcal{E}(\pi) \leq l + 2 \\ +\infty, & \text{otherwise} \end{cases} \geq \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \mathcal{E}(\pi) < \infty \\ +\infty, & \text{otherwise} \end{cases}.$$

Finally, taking the limit when $\varepsilon \downarrow 0$ in the right hand side of above, it yields

$$\overline{\lim}_{\varepsilon \downarrow 0} \begin{cases} \hat{J}_H(\pi * \iota_\varepsilon), & \text{if } \mathcal{E}(\pi) < \infty \\ +\infty, & \text{otherwise} \end{cases} = J_H(\pi),$$

where we have used that, for $\pi \in \{\pi; \mathcal{E}(\pi) < \infty\}$ it holds that $\pi_t(du) = \rho_t(u)du$, where ρ has well-defined left and right side limits around zero. \square

Proposition 5.13 (Upper bound for compact sets). *For every \mathcal{K} compact subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{K}] \leq - \inf_{\pi \in \mathcal{K}} I(\pi).$$

Proof. Proposition 5.12 can be restated in the form

$$\lim_{\varepsilon \downarrow 0} \lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} \lim_{\zeta \downarrow 0} \lim_{\gamma \downarrow 0} \lim_{j \rightarrow \infty} \lim_{m \rightarrow \infty} -J_{H, \gamma, \varepsilon, \zeta}^{k, l, m, j}(\pi) \leq -J_H(\pi),$$

for all $\pi \in \mathcal{D}_{\mathcal{M}}$. Plugging this into (5.15) leads to

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{K}] \leq \sup_{\pi \in \mathcal{K}} \inf_H \{-J_H(\pi)\} = - \inf_{\pi \in \mathcal{K}} \sup_H J_H(\pi) = - \inf_{\pi \in \mathcal{K}} I(\pi).$$

\square

5.3. Upper bound for closed sets

Proposition 5.14 (Upper bound for closed sets). *For every \mathcal{C} closed subset of $\mathcal{D}_{\mathcal{M}}$,*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{C}] \leq - \inf_{\pi \in \mathcal{C}} I(\pi).$$

By exponential tightness, we mean that there exists compact sets $K_n \subset \mathcal{D}_{\mathcal{M}}$ such that

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[K_n^c] \leq -n, \quad \forall n \in \mathbb{N}.$$

It is well known that the upper bound for closed sets is an immediate consequence of upper bound for compact sets plus exponential tightness. We include the proposition below for sake of completeness.

Proposition 5.15. *If the sequence of probabilities $\{\mathbb{Q}_{\mu_N}\}_{N \geq 1}$ is exponentially tight and the inequality*

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[K] \leq - \inf_{\pi \in K} I(\pi)$$

holds for any compact set K , then $\{\mathbb{Q}_{\mu_N}\}_{N \geq 1}$ satisfies

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[C] \leq - \inf_{\pi \in C} I(\pi),$$

for any closed set C .

Proof. Let C be a closed set. Since $\mathbb{Q}_{\mu_N} C] \leq \mathbb{Q}_{\mu_N} [C \cap K_n] + \mathbb{Q}_{\mu_N} [K_n^c]$ and $C \cap K_n$ is compact,

$$\begin{aligned} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [C] &\leq \max \left\{ \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [C \cap K_n], \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [K_n^c] \right\} \\ &\leq \max \left\{ - \inf_{\pi \in C \cap K_n} I(\pi), -n \right\} \leq \max \left\{ - \inf_{\pi \in C} I(\pi), -n \right\}. \end{aligned}$$

Since n is arbitrary, the inequality follows. \square

The rest of this section is concerned about exponential tightness, which we claim it is a consequence of next lemma:

Lemma 5.16. *For all $\varepsilon > 0$, $\delta > 0$ and $H \in C^2(\mathbb{T})$, denote*

$$\mathcal{C}_{H,\delta,\varepsilon} := \left\{ \pi \in \mathcal{D}_{\mathcal{M}}; \sup_{s \leq t \leq s+\delta} |\langle \pi_t, H_\ell \rangle - \langle \pi_s, H_\ell \rangle| \leq \varepsilon, \forall s \in [0, T] \right\}.$$

Then, for every $\varepsilon > 0$ and every function $H \in C^2(\mathbb{T})$, holds

$$\lim_{\delta \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [\pi \notin \mathcal{C}_{H,\delta,\varepsilon}] = -\infty.$$

Indeed, suppose the statement above. Let $\{H_l\}_\ell \subset C^2(\mathbb{T})$ be a dense set of functions in $C(\mathbb{T})$ for the uniform topology. For each $\delta > 0$ and $\ell, m \in \mathbb{N}$, denote by $C_{\ell,\delta,\frac{1}{m}}$ the set $\mathcal{C}_{H_\ell,\delta,\varepsilon}$, with $\varepsilon = \frac{1}{m}$. Then the Lemma 5.16 can be reformulated as:

$$\lim_{\delta \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [\pi \notin C_{\ell,\delta,\frac{1}{m}}] = -\infty, \quad \forall \ell, m \geq 1.$$

Claim: For all positive integers ℓ, m, n , there exists $\tilde{\delta} = \tilde{\delta}(\ell, m, n) > 0$ such that

$$\mathbb{Q}_{\mu_N} [\pi \notin C_{\ell,\tilde{\delta},\frac{1}{m}}] \leq e^{-N n m \ell}, \quad \forall N \in \mathbb{N}.$$

In fact, fixed the positive integers ℓ, m , in view of above, for all $n \in \mathbb{N}$ there exists $\delta_0 = \delta_0(\ell, m, n) > 0$ such that

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [\pi \notin C_{\ell,\delta,\frac{1}{m}}] \leq -n \ell m, \quad \forall \delta \in (0, \delta_0].$$

Then, for each $\delta \in (0, \delta_0]$ there exist $N_\delta = N_\delta(\delta, \ell, m, n) \in \mathbb{N}$ such that

$$\mathbb{Q}_{\mu_N} [\pi \notin C_{\ell,\delta,\frac{1}{m}}] \leq e^{-N n m \ell}, \quad \forall N \geq N_\delta.$$

Since for $0 < \delta_1 < \delta_2$

$$C_{\ell,\delta_2,\frac{1}{m}} \subseteq C_{\ell,\delta_1,\frac{1}{m}},$$

then

$$[\pi \notin C_{\ell,\delta_1,\frac{1}{m}}] \subseteq [\pi \notin C_{\ell,\delta_2,\frac{1}{m}}], \quad \text{for } 0 < \delta_1 < \delta_2. \quad (5.20)$$

Then, denoting $N_0 = N_{\delta_0}(\ell, m, n)$ (which depends only on ℓ, m, n , because δ_0 is a function of ℓ, m, n), we have

$$\mathbb{Q}_{\mu_N} \left[\pi \notin C_{\ell, \delta, \frac{1}{m}} \right] \leq \mathbb{Q}_{\mu_N} \left[\pi \notin C_{\ell, \delta_0, \frac{1}{m}} \right] \leq e^{-N n m \ell}, \quad (5.21)$$

$\forall \delta \in (0, \delta_0]$ and $\forall N \geq N_0$. More one observation is that, for $\ell, m \in \mathbb{N}$ fixed, $C_{\ell, \delta, \frac{1}{m}} \nearrow \mathcal{D}_{\mathcal{M}}$, as $\delta \searrow 0$, it is true because the set $\mathcal{D}_{\mathcal{M}}$ is composed of càdlàg trajectories. Since the $[\pi \notin C_{\ell, \delta, \frac{1}{m}}]$ goes to empty set, when $\delta \rightarrow 0$, then for N fixed the probability $\mathbb{Q}_{\mu_N}[\pi \notin C_{\ell, \delta_N, \frac{1}{m}}]$ goes to zero, as $\delta \rightarrow 0$. Therefore, for each fixed $N \in \mathbb{N}$, we can choose

$$\tilde{\delta}_N = \tilde{\delta}_N(\ell, m, n) \leq \delta_0(\ell, m, n) = \delta_0 \quad (5.22)$$

such that

$$\mathbb{Q}_{\mu_N} \left[\pi \notin C_{\ell, \tilde{\delta}_N, \frac{1}{m}} \right] \leq e^{-N n m \ell}. \quad (5.23)$$

Then, denote by

$$\tilde{\delta} := \min_{N < N_0} \tilde{\delta}_N \leq \delta_0.$$

Let $N \in \mathbb{N}$. If $N < N_0$, then, by $\tilde{\delta} \leq \tilde{\delta}_N$, (5.20) and (5.23), we have

$$\mathbb{Q}_{\mu_N} \left[\pi \notin C_{\ell, \tilde{\delta}, \frac{1}{m}} \right] \leq \mathbb{Q}_{\mu_N} \left[\pi \notin C_{\ell, \tilde{\delta}_N, \frac{1}{m}} \right] \leq e^{-N n m \ell}.$$

And, if $N \geq N_0$, then, by the construction $\tilde{\delta} \leq \delta_0$ (see (5.22)) and (5.21), we get

$$\mathbb{Q}_{\mu_N} \left[\pi \notin C_{\ell, \tilde{\delta}, \frac{1}{m}} \right] \leq e^{-N n m \ell}.$$

Thus, it finishes the proof of the claim.

Now, define

$$K_n = \bigcap_{l \geq 1, m \geq 1} C_{\ell, \tilde{\delta}, \frac{1}{m}}.$$

In order to prove that K_n is a compact set for each $n \geq 1$, we use a similar fashion of the Arzelà-Ascoli theorem, which says that a set of functions $K_n \subset \mathcal{D}_{\mathcal{M}}$ is relatively compact if it is uniformly bounded, and

$$\lim_{\delta \rightarrow 0} \sup_{\pi \in K_n} \omega'_\delta(\pi) = 0, \quad (5.24)$$

where

$$\omega'_\delta(\pi) := \inf_{\{t_i\}} \max_i \sup_{s, t \in [t_{i-1}, t_i)} d(\pi_s, \pi_t),$$

where the infimum is taken over all meshes $0 = t_0 < t_1 < \dots < t_r$ with $t_i - t_{i-1} > \delta$ and d is the metric on \mathcal{M} . The expression $\omega'_\delta(\pi)$ is the so called modulus of continuity of the $\mathcal{D}_{\mathcal{M}}$. We start by observing that K_n is uniformly

bounded, because $K_n \subset \mathcal{D}_{\mathcal{M}}$ (recall the Definition of \mathcal{M} in (2.4)). The limit (5.24) is a consequence of the limit

$$\lim_{\delta \rightarrow 0} \sup_{\pi \in K_n} \omega_{\delta}(\pi) = 0, \quad (5.25)$$

where

$$\omega_{\delta}(\pi) := \sup_{|t-s| \leq \delta} d(\pi_s, \pi_t).$$

To prove this limit we start by recalling the metric d on \mathcal{M} :

$$d(\pi_s, \pi_t) = \sum_{\ell} \frac{1}{2^{\ell}} \frac{|\langle \pi_t, H_{\ell} \rangle - \langle \pi_s, H_{\ell} \rangle|}{1 + |\langle \pi_t, H_{\ell} \rangle - \langle \pi_s, H_{\ell} \rangle|} \leq \sum_{\ell} \frac{1}{2^{\ell}} |\langle \pi_t, H_{\ell} \rangle - \langle \pi_s, H_{\ell} \rangle|.$$

Thus, if $|t-s| \leq \tilde{\delta}$, as we can suppose, without loss of generality, that $s \leq t$, then $s \leq t \leq \tilde{\delta} + s$. Thus, for $\pi \in K_n$, we have

$$|\langle \pi_t, H_{\ell} \rangle - \langle \pi_s, H_{\ell} \rangle| \leq \frac{1}{m},$$

for all $\ell, m \in \mathbb{N}$. Therefore, for each $\pi \in K_n$, $d(\pi_s, \pi_t) \leq \frac{1}{m}$, for all $m \in \mathbb{N}$. As consequence

$$\sup_{\pi \in K_n} \omega_{\tilde{\delta}}(\pi) \leq \frac{1}{m},$$

for all $m \in \mathbb{N}$. Since $\delta \mapsto \omega_{\delta}(\pi)$ is decreasing on δ (for π fixed), for all $\delta \leq \tilde{\delta}$ and $\pi \in K_n$, we have $\omega_{\delta}(\pi) \leq \omega_{\tilde{\delta}}(\pi) \leq \sup_{\pi \in K_n} \omega_{\tilde{\delta}}(\pi) \leq \frac{1}{m}$, for all $m \in \mathbb{N}$. Then, $\sup_{\pi \in K_n} \omega_{\delta}(\pi) \leq \frac{1}{m}$, for all $\delta \leq \tilde{\delta}$ and $m \in \mathbb{N}$. Therefore, the limit (5.25) follows. Since K_n is a closed set, we have proved that K_n is a compact set.

Furthermore, by construction of the set K_n and the last claim, we have

$$\mathbb{Q}_{\mu_N} [\pi \notin K_n] \leq \sum_{\substack{\ell \geq 1 \\ m \geq 1}} e^{-Nnm\ell} \leq C e^{-Nn},$$

where C is a constant not depending in the parameters. In particular,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [\pi \notin K_n] \leq -n,$$

which is the exponential tightness. Therefore, our goal from now on is to prove Lemma 5.16.

Proof of Lemma 5.16. Fix $\varepsilon > 0$ and $H \in C^2(\mathbb{T})$. Recalling the definition of the set $\mathcal{C}_{H,\delta,\varepsilon}$, then the set $[\pi \notin \mathcal{C}_{H,\delta,\varepsilon}]$ is equal to

$$\left\{ \pi \in \mathcal{D}_{\mathcal{M}}; \sup_{s \leq t \leq s+\delta} |\langle \pi_t, H \rangle - \langle \pi_s, H \rangle| > \varepsilon, \text{ for some } s \in [0, T] \right\}.$$

Consider the partition of the interval $[0, T]$ with the size of the mesh equal to δ , then there exists $k \in \{0, \dots, \lfloor T\delta^{-1} \rfloor\}$ such that $k\delta \leq s < (k+1)\delta$. Thus,

$$\{t \in [0, T]; s \leq t \leq s + \delta\} = \Gamma_s^1 \cup \Gamma_s^2,$$

where

$$\Gamma_s^1 = \{t \in [0, T]; s \leq t \leq (k+1)\delta\} \text{ and } \Gamma_s^2 = \{t \in [0, T]; (k+1)\delta \leq t \leq s + \delta\}.$$

Since

$$\begin{aligned} & \sup_{s \leq t \leq s+\delta} |\langle \pi_t, H \rangle - \langle \pi_s, H \rangle| \\ & \leq \sup_{t \in \Gamma_s^1} |\langle \pi_t, H \rangle - \langle \pi_s, H \rangle| + \sup_{t \in \Gamma_s^2} |\langle \pi_t, H \rangle - \langle \pi_s, H \rangle| \\ & \leq \sup_{k\delta \leq t \leq (k+1)\delta} |\langle \pi_t, H \rangle - \langle \pi_{k\delta}, H \rangle| + 2|\langle \pi_s, H \rangle - \langle \pi_{k\delta}, H \rangle| \\ & + |\langle \pi_{(k+1)\delta}, H \rangle - \langle \pi_{k\delta}, H \rangle| + \sup_{(k+1)\delta \leq t \leq (k+2)\delta} |\langle \pi_t, H \rangle - \langle \pi_{(k+1)\delta}, H \rangle| \\ & \leq 4 \sup_{k\delta \leq t \leq (k+1)\delta} |\langle \pi_t, H \rangle - \langle \pi_{k\delta}, H \rangle| \\ & + \sup_{(k+1)\delta \leq t \leq (k+2)\delta} |\langle \pi_t, H \rangle - \langle \pi_{(k+1)\delta}, H \rangle|, \end{aligned}$$

then

$$\left\{ \pi; \sup_{s \leq t \leq s+\delta} |\langle \pi_t, H \rangle - \langle \pi_s, H \rangle| > \varepsilon, \text{ for some } s \in [0, T] \right\} \subseteq \bigcup_{k=0}^{\lfloor T\delta^{-1} \rfloor} A_{k,\delta}^N,$$

where

$$A_{k,\delta}^N = \left\{ \sup_{k\delta \leq t \leq (k+1)\delta} |\langle \pi_t, H \rangle - \langle \pi_{k\delta}, H \rangle| > \varepsilon/5 \right\}.$$

Thus, for all $\delta > 0$,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} \left[\pi \notin \mathcal{C}_{\ell, \delta, \frac{1}{m}} \right] \leq \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \sum_{k=0}^{\lfloor T\delta^{-1} \rfloor} \mathbb{Q}_{\mu_N} [A_{k,\delta}^N].$$

Recalling (3.16), the limit in the right-hand side above is bounded from above by

$$\max_{k \in \{0, \dots, \lfloor T\delta^{-1} \rfloor\}} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [A_{k,\delta}^N] \leq \sum_{k=0}^{\lfloor T\delta^{-1} \rfloor} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [A_{k,\delta}^N].$$

Then, in order to prove the Lemma 5.16, it is enough to show that

$$\lim_{\delta \downarrow 0} \sum_{k=0}^{\lfloor T\delta^{-1} \rfloor} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [A_{k,\delta}^N] = -\infty. \quad (5.26)$$

We begin by observing that $A_{k,\delta}^N = B_{k,\delta}^{H,N} \cup B_{k,\delta}^{-H,N}$, where

$$B_{k,\delta}^{H,N} = \left\{ \sup_{k\delta \leq t \leq (k+1)\delta} \langle \pi_t, H \rangle - \langle \pi_{k\delta}, H \rangle > \varepsilon/10 \right\}.$$

Hence, recalling (3.16), to obtain (5.26) it will be sufficient to assure that

$$\lim_{\delta \downarrow 0} \sum_{k=0}^{\lfloor T\delta^{-1} \rfloor} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [B_{k,\delta}^{H,N}] = -\infty, \quad (5.27)$$

for any $H \in C^2(\mathbb{T})$. To obtain the claim above we will analyse the limit $\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [B_{k,\delta}^{H,N}]$ for fixed δ, k and H . Denote

$$M_t^H = \exp \left\{ N \left[\langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \frac{1}{N} \int_0^t e^{-N \langle \pi_s^N, H \rangle} (\partial_s + N^2 L_N) e^{N \langle \pi_s^N, H \rangle} ds \right] \right\},$$

which is a positive mean one martingale with respect to the natural filtration. By assumption H does not depend on time. Keeping this in mind and performing elementary calculations we get

$$M_t^H = \exp \left\{ N \langle \pi_t^N, H \rangle - N \langle \pi_0^N, H \rangle - \int_0^t U_N(H, s, \eta_s) ds \right\},$$

where

$$U_N(H, s, \eta_s) = \sum_{|x-y|=1} N \xi_{x,y}^N \left\{ e^{[H(\frac{x}{N}) - H(\frac{y}{N})]} - 1 \right\} \eta_s(x) (1 - \eta_s(y))$$

with

$$\xi_{x,y}^N = \xi_{y,x}^N = \begin{cases} N^{-1}, & \text{if } x = -1 \text{ and } y = 0, \\ 0, & \text{if } |x - y| > 1, \\ 1, & \text{otherwise.} \end{cases}$$

Notice that $\{M_t^H / M_{k\delta}^H\}_{t \geq k\delta}$ is also a positive mean one martingale. Adding and subtracting the integral part, we get

$$\mathbb{Q}_{\mu_N} [B_{k,\delta}^{H,N}] \leq \mathbb{Q}_{\mu_N} [C_{k,\delta}^N] + \mathbb{Q}_{\mu_N} [D_{k,\delta}^N],$$

where

$$C_{k,\delta}^N = \left\{ \sup_{k\delta \leq t \leq (k+1)\delta} \frac{1}{N} \log \left(\frac{M_t^H}{M_{k\delta}^H} \right) > \varepsilon/20 \right\}$$

and

$$D_{k,\delta}^N = \left\{ \sup_{k\delta \leq t \leq (k+1)\delta} \int_{k\delta}^t U_N(H, s, \eta_s) ds > \varepsilon/20 \right\}.$$

From the above and again (3.16), we have that

$$\begin{aligned} & \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [B_{k,\delta}^{H,N}] \\ & \leq \max \left\{ \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [C_{k,\delta}^N], \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [D_{k,\delta}^N] \right\}, \end{aligned} \quad (5.28)$$

for all $\delta > 0$ and $k \in \{0, 1, \dots, \lfloor T\delta^{-1} \rfloor\}$. Since $H \in C^2(\mathbb{T})$, by a Taylor expansion it is easy to verify that $|\int_{k\delta}^t U_N(H, s, \eta_s) ds|$ is bounded by $C(H)\delta$, for all $t \in [k\delta, (k+1)\delta]$. Thus, if we take $\delta \in (0, \tilde{C})$ with $\tilde{C} = \varepsilon/(20C(H))$, then $\mathbb{Q}_{\mu_N}[D_{k,\delta}^N] = 0$ for all $k \in \{0, 1, \dots, \lfloor T\delta^{-1} \rfloor\}$, and therefore the inequality (5.28) becomes

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [B_{k,\delta}^{H,N}] \leq \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [C_{k,\delta}^N],$$

provided $\delta < \tilde{C}$. Now we handle the set $C_{k,\delta}^N$ in the way

$$\begin{aligned} \mathbb{Q}_{\mu_N} [C_{k,\delta}^N] &= \mathbb{Q}_{\mu_N} \left[\sup_{k\delta \leq t \leq (k+1)\delta} \frac{1}{N} \log \left(\frac{M_t^H}{M_{k\delta}^H} \right) > \varepsilon/20 \right] \\ &= \mathbb{Q}_{\mu_N} \left[\sup_{k\delta \leq t \leq (k+1)\delta} \frac{M_t^H}{M_{k\delta}^H} > e^{\varepsilon N/20} \right] \leq \frac{1}{e^{\varepsilon N/20}}, \end{aligned}$$

where in last inequality we have used Doob's inequality since $\{M_t^H/M_{k\delta}^H\}_{t \geq k\delta}$ is a mean one positive martingale. Thus,

$$\overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [B_{k,\delta}^{H,N}] \leq -\varepsilon/20,$$

for all $\delta > 0$, $k = 0, 1, \dots, \lfloor T\delta^{-1} \rfloor$, and $H \in C^2(\mathbb{T})$. By the inequality above,

$$\lim_{\delta \downarrow 0} \sum_{k=0}^{\lfloor T\delta^{-1} \rfloor} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N} [B_{k,\delta}^{H,N}] \leq \lim_{\delta \downarrow 0} \frac{-\varepsilon}{20} (\lfloor T\delta^{-1} \rfloor + 1) = -\infty,$$

implying (5.27), which finishes the proof. \square

6. Large deviations lower bound for smooth profiles

Next, we obtain a non-variational formulation of the rate functional I for profiles ρ whose are solutions of the hydrodynamical equation for some perturbation $H \in C^{1,2}([0, T] \times [0, 1])$.

Proposition 6.1. *Given $H \in C^{1,2}([0, T] \times [0, 1])$, let ρ^H be the unique weak solution of (2.11). Then,*

$$\begin{aligned} I(\rho^H) &:= \sup_G \hat{J}_G(\rho^H) = \hat{J}_H(\rho^H) \\ &= \int_0^T \langle \chi(\rho_t^H), (\partial_u H_t)^2 \rangle dt + \int_0^T \rho_t^H(0^-)(1 - \rho_t^H(0^+)) \Gamma(\delta H_t(0)) dt \\ &\quad + \int_0^T \rho_t^H(0^+)(1 - \rho_t^H(0^-)) \Gamma(-\delta H_t(0)) dt, \end{aligned} \quad (6.1)$$

where $\Gamma(y) = 1 - e^y + y e^y$, $\forall y \in \mathbb{R}$.

Although of quite simple proof, this result has a deep interpretation. The functional $-\hat{J}_G(\rho)$ has the meaning of being *the price* to observe the profile ρ when we perturb the system by G . The equality $\sup_G \hat{J}_G(\rho^H) = \hat{J}_H(\rho^H)$ says that the minimum cost to observe the profile ρ is reached by picking up the perturbation $G = H$, where H is such that $\rho = \rho^H$, i.e., such that ρ is a solution of (2.11).

Proof. Replacing the integral equation (2.13) in the definition of \hat{J} given in (2.15), we get

$$\begin{aligned} \hat{J}_G(\rho^H) &= \int_0^T \langle \chi(\rho_t^H), (\partial_u H_t)^2 \rangle dt - \int_0^T \langle \chi(\rho_t^H), (\partial_u H_t - \partial_u G_t)^2 \rangle dt \\ &\quad + \int_0^T \rho_t^H(0^-)(1 - \rho_t^H(0^+)) \bar{\Gamma}(\delta G_t(0), \delta H_t(0)) dt \\ &\quad + \int_0^T \rho_t^H(0^+)(1 - \rho_t^H(0^-)) \bar{\Gamma}(-\delta G_t(0), -\delta H_t(0)) dt, \end{aligned}$$

where $\bar{\Gamma}(x, y) = 1 - e^x + x e^y$, $\forall x, y \in \mathbb{R}$. Let $y \in \mathbb{R}$ fixed. The function $x \mapsto \bar{\Gamma}(x, y)$ assumes its maximum at $x = y$. Therefore, $I(\rho^H) = \sup_G \hat{J}_G(\rho^H) = \hat{J}_H(\rho^H)$. Noticing that $\Gamma(y) = \bar{\Gamma}(y, y)$ we arrive at (6.1). \square

Remark 6.2. As natural, if λ is the unique weak solution of (2.6), then the rate functional vanishes at λ . In fact, given $G \in C^{1,2}([0, T] \times [0, 1])$, we have $\ell_G(\lambda) = 0$ because λ satisfies the integral equation (2.7). Since $\psi(u) = e^u - u - 1 \geq 0$, it yields $\hat{J}_G(\lambda) \leq 0$. And $\hat{J}_G(\lambda) = 0$ if G is constant.

By Proposition 6.1, profiles that are solution of (2.11) for some H provides a special representation for the rate functional. This motivates the next definition.

Definition 6.3. Denote by $\mathcal{D}_{\mathcal{M}_0}^{\text{eq}}$ the subset of $\mathcal{D}_{\mathcal{M}_0}$ consisting of all paths $\pi_t(du) = \rho_t(u) du$ for which there exists some $H \in C^{1,2}([0, T] \times [0, 1])$ such that $\rho = \rho^H$ is the unique weak solution of (2.11).

We begin by proving the lower bound for trajectories in $\mathcal{D}_{\mathcal{M}_0}^{\text{eq}}$. In the following we present the lower bound in the set of smooth trajectories, $\mathcal{D}_{\mathcal{M}_0}^{\mathcal{S}}$.

Proposition 6.4. Let \mathcal{O} be an open set of $\mathcal{D}_{\mathcal{M}}$. Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq - \inf_{\pi \in \mathcal{O} \cap \mathcal{D}_{\mathcal{M}_0}^{\text{eq}}} I(\pi).$$

Proof. This proof is essentially the same as that found in [KL]. Fix the open set \mathcal{O} . Given $\pi \in \mathcal{O} \cap \mathcal{D}_{\mathcal{M}_0}^{\text{eq}}$, by definition there exists $H \in C^{1,2}([0, T] \times [0, 1])$ such that $\pi_t(du) = \rho_t^H(u) du$, where ρ^H is the weak solution of (2.11). Denote by $\mathbb{P}_{\mu_N}^{H, \mathcal{O}}$ the probability on the space \mathcal{D}_{Ω_N} defined by

$$\mathbb{P}_{\mu_N}^{H, \mathcal{O}}[A] = \frac{\mathbb{P}_{\mu_N}^H[A, \pi^N \in \mathcal{O}]}{\mathbb{P}_{\mu_N}^H[\pi^N \in \mathcal{O}]},$$

for any A measurable subset of \mathcal{D}_{Ω_N} . Within this definition,

$$\frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] = \frac{1}{N} \log \mathbb{E}_{\mu_N}^{H, \mathcal{O}} \left[\frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}^H} \right] + \frac{1}{N} \log \mathbb{Q}_{\mu_N}^H[\mathcal{O}].$$

Since \mathcal{O} is an open set that contains ρ^H , by the Proposition 4.1 the second term in the right hand side of above converges to zero as N increases to infinity. Since the logarithm is a concave function, by Jensen's inequality the first term in the right hand side of above is bounded from below by

$$\mathbb{E}_{\mu_N}^{H, \mathcal{O}} \left[\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}^H} \right].$$

Adding and subtracting the indicator function of the set $\{\pi^N \in \mathcal{O}^c\}$, the last expectation becomes

$$\frac{1}{\mathbb{Q}_{\mu_N}^H[\mathcal{O}]} \left\{ -\frac{1}{N} \mathbf{H}(\mathbb{P}_{\mu_N}^H | \mathbb{P}_{\mu_N}) - \mathbb{E}_{\mu_N}^H \left[\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}^H} \mathbf{1}_{\{\pi^N \in \mathcal{O}^c\}} \right] \right\}, \quad (6.2)$$

where

$$\mathbf{H}(\mathbb{P}_{\mu_N}^H | \mathbb{P}_{\mu_N}) := \mathbb{E}_{\mu_N}^H \left[\log \frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}} \right] = -\mathbb{E}_{\mu_N}^H \left[\log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}^H} \right] \quad (6.3)$$

is the so-called relative entropy of $\mathbb{P}_{\mu_N}^H$ with respect to \mathbb{P}_{μ_N} . Again by Proposition 4.1 we have that $\mathbb{Q}_{\mu_N}^H[\mathcal{O}]$ converges to one as N increases to infinity. By (4.6) the expression $\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}_{\mu_N}}{\mathbf{d}\mathbb{P}_{\mu_N}^H}$ is bounded, hence the second term inside braces in (6.2) vanishes as N increases to ∞ . Thus

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq \lim_{N \rightarrow \infty} -\frac{1}{N} \mathbf{H}(\mathbb{P}_{\mu_N}^H | \mathbb{P}_{\mu_N}) = -I(\rho^H),$$

where the last equality has an importance for itself and for this reason it is postponed to the Lemma 6.5 proved next. \square

Lemma 6.5. *Let $H \in C^{1,2}([0, T] \times [0, 1])$. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{H}(\mathbb{P}_{\mu_N}^H | \mathbb{P}_{\mu_N}) = I(\rho^H),$$

where ρ^H is the unique weak solution of (2.11).

Proof. Using the formula (6.3) for the relative entropy, we get

$$\frac{1}{N} \mathbf{H}(\mathbb{P}_{\mu_N}^H | \mathbb{P}_{\mu_N}) = \frac{1}{N} \mathbb{E}_{\mu_N}^H \left[\log \frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}} \mathbf{1}_{B_{\delta, \varepsilon}^H} \right] + \frac{1}{N} \mathbb{E}_{\mu_N}^H \left[\log \frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}} \mathbf{1}_{(B_{\delta, \varepsilon}^H)^c} \right], \quad (6.4)$$

where the set $B_{\delta, \varepsilon}^H$ was defined in (5.3). We claim that the event $(B_{\delta, \varepsilon}^H)^c$ is superexponentially small with respect to $\mathbb{P}_{\mu_N}^H$. Indeed, by (4.6) we have

$$\mathbb{P}_{\mu_N}^H \left[(B_{\delta, \varepsilon}^H)^c \right] = \mathbb{E}_{\mu_N} \left[\frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}} \mathbf{1}_{(B_{\delta, \varepsilon}^H)^c} \right] \leq e^{C(H, T)N} \mathbb{P}_{\nu_\alpha^N} \left[(B_{\delta, \varepsilon}^H)^c \right]$$

and then by (5.4) we get

$$\overline{\lim}_{\varepsilon \downarrow 0} \overline{\lim}_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}^H \left[(B_{\delta, \varepsilon}^H)^c \right] = -\infty.$$

Provided by the limit above and the fact that $\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}}$ is bounded, the right hand side of (6.4) is

$$\frac{1}{N} \mathbb{E}_{\mu_N}^H \left[\log \frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}} \mathbf{1}_{B_{\delta, \varepsilon}^H} \right] + o_N(1), \quad (6.5)$$

for all $\delta > 0$ and each small enough $\varepsilon = \varepsilon(\delta)$. Applying the expression (5.9) for the Radon-Nikodym derivative, $\frac{1}{N} \log \frac{\mathbf{d}\mathbb{P}_{\mu_N}^H}{\mathbf{d}\mathbb{P}_{\mu_N}}$ on the set $B_{\delta, \varepsilon}^H$ is equal to

$$\hat{J}_H((\pi^N * \iota_\gamma^s) * \iota_\varepsilon) + O_{H, T, \varepsilon, \gamma}(\frac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\frac{\gamma}{\varepsilon}),$$

for all $\delta > 0$ and all ε and γ small enough. Since this expression is bounded and the probability of $(B_{\delta, \varepsilon}^H)^c$ with respect to $\mathbb{P}_{\mu_N}^H$ vanishes as N increases to infinity, the expression (6.5) becomes

$$\mathbb{E}_{\mu_N}^H \left[\hat{J}_H((\pi^N * \iota_\gamma^s) * \iota_\varepsilon) \right] + O_{H, T, \varepsilon, \gamma}(\frac{1}{N}) + O(\delta) + O_H(\varepsilon) + O_H(\frac{\gamma}{\varepsilon}) + o_N(1),$$

for all $\delta > 0$ and all ε and γ small enough. For fixed ε and γ , the map $\rho \mapsto \hat{J}_H((\rho * \iota_\gamma^s) * \iota_\varepsilon)$ is continuous with respect to the Skorohod topology, see the Proposition 5.11. Moreover, by Proposition 4.1 the sequence $\mathbb{Q}_{\mu_N}^H$ converges weakly to the probability concentrated on the weak solution of (2.11). In particular, as N increases to infinity, the previous expectation converges to

$$\hat{J}_H((\rho^H * \iota_\gamma^s) * \iota_\varepsilon) + O(\delta) + O_H(\varepsilon) + O_H(\frac{\gamma}{\varepsilon}).$$

Letting $\gamma \downarrow 0$, then taking $\varepsilon \downarrow 0$, finally $\delta \downarrow 0$ and then invoking Lemma 6.1 concludes the proof. \square

Since weak solutions of (2.11) for some H implies the special representation (6.1) for the rate functional, it is natural to study in what conditions a profile ρ can be written as a solution of (2.11). This is the content of the next proposition. Notice that the first equation in (6.6) ahead is nothing else than the partial differential equation (2.11) rearranged.

Proposition 6.6. *Let $\rho \in C^{1,2}([0, T] \times [0, 1])$ such that $0 < \varepsilon \leq \rho \leq 1 - \varepsilon$, for some $\varepsilon > 0$. Then, there exists a unique (strong) solution $H \in C^{1,2}([0, T] \times [0, 1])$ of the elliptic equation*

$$\begin{cases} \partial_u^2 H_t(u) + \frac{\partial_u(\chi(\rho_t(u)))}{\chi(\rho_t(u))} \partial_u H_t(u) = \frac{\Delta \rho_t(u) - \partial_t \rho_t(u)}{2\chi(\rho_t(u))}, \forall u \in (0, 1) \\ \partial_u H_t(0) = \frac{1}{2\chi(\rho_t(0))} [Be^{\delta H_t(0)} - Ce^{-\delta H_t(0)} + \partial_u \rho_t(0)] \\ \partial_u H_t(1) = \frac{1}{2\chi(\rho_t(1))} [Be^{\delta H_t(0)} - Ce^{-\delta H_t(0)} + \partial_u \rho_t(1)] \\ H_t(0) = 0 \end{cases} \quad (6.6)$$

where $B = B(\rho_t) = \rho_t(1)(1 - \rho_t(0))$ and $C = C(\rho_t) = \rho_t(0)(1 - \rho_t(1))$, for all $t \in [0, T]$. Above we are denoting $0 = 0^+$ and $1 = 0^-$.

Proof. For fixed time, the first equation in (6.6) is a linear second order ordinary differential equation in H . The only work is to adjust the solution to satisfy the boundary conditions. Let $z_0 \in \mathbb{R}$ be the unique solution of the transcendental equation $z = (Be^{-z} - Ce^z)\alpha + A$, where

$$\alpha = \alpha(\rho_t) := \int_0^1 \frac{1}{2\chi(\rho_t(v))} dv,$$

$$A = A(\rho_t) := \int_0^1 \frac{\partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w) dw}{2\chi(\rho_t(v))} dv,$$

and $B > 0$ and $C > 0$ are those ones in the statement of the proposition. Let

$$H_t(u) := (Be^{-z_0} - Ce^{z_0}) \int_0^u \frac{1}{2\chi(\rho_t(v))} dv + \int_0^u \frac{\partial_u \rho_t(v) - \partial_t \int_0^v \rho_t(w) dw}{2\chi(\rho_t(v))} dv,$$

for all $t \in [0, T]$. It can be directly checked that H is the solution of (6.6). \square

Recalling the definition of $\mathcal{D}_{\mathcal{M}_0}^S$ given in the Theorem 2.14 and the definition of $\mathcal{D}_{\mathcal{M}_0}^{\text{eq}}$, Proposition 6.6 can be resumed as:

Corollary 6.7. *The set*

$$\mathcal{D}_{\mathcal{M}_0}^S \cap \{\pi \in \mathcal{D}_{\mathcal{M}}; \pi_t(du) = \rho_t(u)du, \text{ with } \varepsilon \leq \rho \leq 1 - \varepsilon \text{ for some } \varepsilon > 0\}$$

is contained in $\mathcal{D}_{\mathcal{M}_0}^{\text{eq}}$.

Despite not convex in general, the rate functional I obtained in our model is convex in some sense. This is subject of the next proposition, to be used in a density argument.

Proposition 6.8. *Let $\rho, \lambda \in \mathcal{D}_{\mathcal{M}}$ with $I(\rho)$ and $I(\lambda)$ finite such that $(\rho_t(0^+) - \lambda_t(0^+))(\rho_t(0^-) - \lambda_t(0^-)) \geq 0$, almost surely in $t \in [0, T]$. Then, for $\theta \in [0, 1]$,*

$$I(\theta\rho + (1 - \theta)\lambda) \leq \theta I(\rho) + (1 - \theta) I(\lambda). \quad (6.7)$$

Proof. Let $\theta \in [0, 1]$. We claim that

$$\hat{J}_H(\theta\rho + (1-\theta)\lambda) \leq \theta\hat{J}_H(\rho) + (1-\theta)\hat{J}_H(\lambda), \quad (6.8)$$

for any $H \in C^{1,2}([0, T] \times [0, 1])$. Recall that $\hat{J}_H(\rho)$ is the sum of linear part in ρ , namely

$$\ell_H(\rho) - \int_0^T \left\{ \rho_t(0^-) \psi(\delta H_t(0)) + \rho_t(0^+) \psi(-\delta H_t(0)) \right\} dt,$$

plus a convex part in ρ , namely $-\int_0^T \langle \chi(\rho_t), (\partial_u H_t)^2 \rangle dt$, and

$$\int_0^T \rho_t(0^-) \rho_t(0^+) \left\{ \psi(\delta H_t(0)) + \psi(-\delta H_t(0)) \right\} dt, \quad (6.9)$$

wherefore we only need to care about this last term. Since $\psi(x) = e^x - x - 1 \geq 0$, we have that $\psi(\delta H_t(0)) + \psi(-\delta H_t(0)) \geq 0$. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by $f(x, y) = xy$. If (x_1, y_1) and (x_2, y_2) are two points of \mathbb{R}^2 such that $(x_2 - x_1)(y_2 - y_1) \geq 0$, then

$$f(\theta(x_1, y_1) + (1-\theta)(x_2, y_2)) \leq \theta f(x_1, y_1) + (1-\theta)f(x_2, y_2). \quad (6.10)$$

To see this, just note that f is convex along lines of the form $y = ax + b$, provided $a > 0$. The inequality (6.10) applied to (6.9) permits to conclude the inequality (6.8), which in his hand leads to (6.7). \square

Proposition 6.9. *Let $\pi \in \mathcal{D}_{\mathcal{M}}$ with $I(\pi) < \infty$. There exists a sequence $\{\pi^\varepsilon\}_{\varepsilon>0}$ in $\mathcal{D}_{\mathcal{M}_0}$ such that π^ε converges to π in $\mathcal{D}_{\mathcal{M}}$ and $\pi_t^\varepsilon(du) = \rho_t^\varepsilon(u) du$ with $\varepsilon \leq \rho_t^\varepsilon(u) \leq 1 - \varepsilon$. Moreover, $\lim_{\varepsilon \downarrow 0} I(\pi^\varepsilon) \leq I(\pi)$.*

Proof. Let $\pi \in \mathcal{D}_{\mathcal{M}}$ with $I(\pi) < \infty$, then $\pi_t(du) = \rho_t(u) du$ and $0 \leq \rho \leq 1$. Consider $\tilde{1}(t, u) = 1$ and $\tilde{0}(t, u) = 0$, for all $t \in [0, T]$ and $u \in \mathbb{T}$. Define $\rho^\varepsilon = \varepsilon \tilde{1} + (1 - 2\varepsilon)\rho + \varepsilon \tilde{0}$ and $\pi_t^\varepsilon(du) = \rho_t^\varepsilon(u) du$. By Lemma 6.8, $I(\pi^\varepsilon) \leq \varepsilon I(\tilde{1}) + (1 - 2\varepsilon)I(\rho) + \varepsilon I(\tilde{0})$. Hence $\lim_{\varepsilon \downarrow 0} I(\pi^\varepsilon) \leq I(\pi)$. \square

We are in position to prove the lower bound for smooth profiles.

Proof of the Theorem 2.14, item (ii). Fix $\pi \in \mathcal{D}_{\mathcal{M}_0}^S \cap \mathcal{O}$ and consider the sequence $\pi_t^\varepsilon(du) = \rho_t^\varepsilon(u) du$, where $\rho_t^\varepsilon(u) = \varepsilon + (1 - 2\varepsilon)\rho_t(u)$, as in the proof of the Proposition 6.9. That is, such that $\varepsilon < \rho^\varepsilon < 1 - \varepsilon$ with $\rho^\varepsilon \in C^{1,2}([0, T] \times [0, 1])$. By Corollary 6.7 and since \mathcal{O} is open, we have that $\pi^\varepsilon \in \mathcal{D}_{\mathcal{M}_0}^{\text{eq}} \cap \mathcal{O}$ for small enough $\varepsilon > 0$.

By Proposition 6.4,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq - \inf_{\lambda \in \mathcal{O} \cap \mathcal{D}_{\mathcal{M}_0}^{\text{eq}}} I(\lambda) \geq -I(\pi^\varepsilon).$$

Taking the limit infimum in the right hand side of inequality above and using the Lemma 6.9, we get

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq -\overline{\lim}_{\varepsilon \rightarrow 0} I(\pi^\varepsilon) \geq -I(\pi).$$

Since π is an arbitrary trajectory on the set $\mathcal{O} \cap \mathcal{D}_{\mathcal{M}_0}^S$, we can optimize over all elements in this set, obtaining therefore

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{Q}_{\mu_N}[\mathcal{O}] \geq \sup_{\pi \in \mathcal{O} \cap \mathcal{D}_{\mathcal{M}_0}^S} -I(\pi) = -\inf_{\pi \in \mathcal{O} \cap \mathcal{D}_{\mathcal{M}_0}^S} I(\pi),$$

which finishes the proof. \square

Appendix A: Uniqueness of strong solutions

As aforementioned, we have assumed uniqueness of weak solutions of (2.11), a delicate problem in the area of partial differential equations for which we have no argument. In this appendix we present uniqueness of strong solutions of (2.11).

Theorem A.1. *Let $\rho_0 : \mathbb{R} \rightarrow [0, 1]$ be measurable profile. Then, there exists at most one strong solution of the partial differential equation (2.11).*

Proof. We will describe a general situation that includes the PDE (2.11). Let u_1 and u_2 two strong solutions of

$$\begin{cases} \partial_t u = \partial_x^2 u + F(t, x, u, \partial_x u) \\ u(0, x) = \bar{u}(x) \\ \partial_x u(0) = H_0(t, x, u(0), u(1)) \\ \partial_x u(1) = H_1(t, x, u(0), u(1)) \end{cases}$$

where F, H_0, H_1 are smooth functions. Let $v = u_1 - u_2$. Hence $v(0, x) = 0$ and $\partial_t v = \partial_x^2 v + L$, where

$$L = F(t, x, u_1, \partial_x u_1) - F(t, x, u_2, \partial_x u_2).$$

By smoothness, there exists a constant $C > 0$ such that hold the estimates

$$\begin{aligned} |F(t, x, u_1, \partial_x u_1) - F(t, x, u_2, \partial_x u_2)| &\leq C(|v| + |\partial_x v|), \\ |H_i(t, x, u_1(0), u_1(1)) - H_i(t, x, u_2(0), u_2(1))| &\leq C(|v(0)| + |v(1)|), \end{aligned}$$

for $i = 0, 1$. An application of Young's inequality implies that, for all $\varepsilon > 0$, there exists $A(\varepsilon) > 0$ such that

$$|v(0)|^2 + |v(1)|^2 \leq \varepsilon \int_0^1 (\partial_x v)^2 dx + A(\varepsilon) \int_0^1 v^2 dx, \quad (\text{A.1})$$

for any time $t > 0$. Define $q(t) = \int_0^1 v^2(t, x) dx$. Then

$$\begin{aligned} q'(t) &= 2 \int_0^1 v \partial_t v dx = 2 \int_0^1 v \partial_x^2 v dx + 2 \int_0^1 v L dx \\ &= 2 v(1) \partial_x v(1) - 2 v(0) \partial_x v(0) - 2 \int_0^1 (\partial_x v)^2 dx + 2 \int_0^1 v L dx. \end{aligned}$$

Thus, by previous estimates,

$$\begin{aligned} q'(t) &\leq C_1 \left((v(0))^2 + |v(0)| |v(1)| + (v(1))^2 \right) - 2 \int_0^1 (\partial_x v)^2 dx \\ &\quad + C_1 \int_0^1 v^2 dx + C_1 \int_0^1 |v| |\partial_x v| dx. \end{aligned}$$

Again by Young's Inequality,

$$\begin{aligned} q'(t) &\leq -2 \int_0^1 (\partial_x v)^2 dx + C_2 \left((v(0))^2 + (v(1))^2 \right) \\ &\quad + C_2 \int_0^1 v^2 dx + \beta \int_0^1 (\partial_x v)^2 dx. \end{aligned}$$

where β can be chosen small as necessary. Recalling (A.1) with small ε gives us

$$q'(t) \leq -\frac{1}{2} \int_0^1 (\partial_x v)^2 dx + C_3 \int_0^1 v^2 dx$$

implying $q'(t) \leq C_3 q(t)$. Noticing that $q(0) = 0$, Gronwall's inequality finishes the proof. \square

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